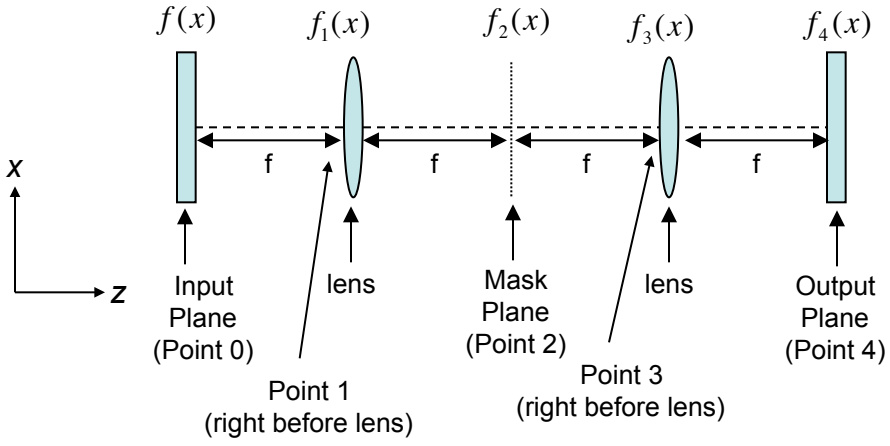


Problem 1 (35marks)

In this problem you will show that the 4-f system produces an inverted image at the output plane with fourier transform of the image occurring at the mask plane by treating a simple plan wave input. You may use the paraxial and frensel approximations and only work in the x direction. With the input wave

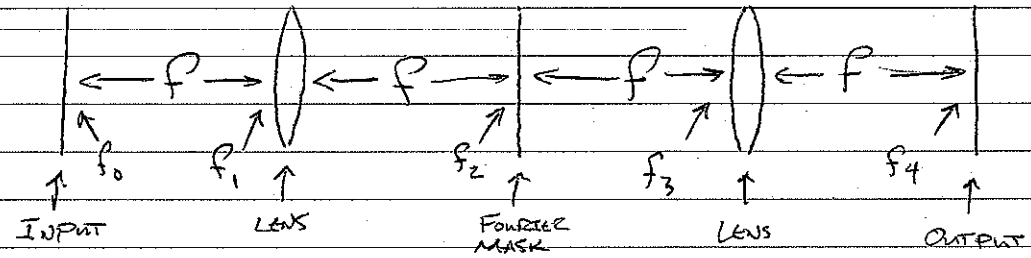
$$U(x,z) = Ae^{i(k_z z + k_x x)} = Ae^{ik(\cos\theta z + \sin\theta x)} \approx Ae^{ik(z+\theta x)}$$

sequentially calculate $f(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ and therefore show that $f_2(x)$ is the fourier transform of $f(x)$ and $f_4(x) = f(-x)$. You will have to carefully keep track of constant phase terms that we usually ignore in lecture. Also describe in words what the image looks like at each point.

Some suggestions:
propagation from point 0 to 1 is easy, you know how a plane wave propagates
point 1 to 2: this should look familiar to you…
point 2 to 3: I would use convolution and the approximated free space impulse response function
point 3 to 4: I can't give the whole answer to you!

PHYS 408 PS 7 SOLUTIONS

(35) (1) WE ARE WORKING WITH A 4f SYSTEM



WE HAVE TWO CONVEX LENSES

$$t(x,y) = h_0 e^{-iK\frac{x^2+y^2}{2f}}$$

FOR A PLANO CONVEX $f \equiv \frac{R}{n-1}$

FOR A DOUBLE CONVEX $f \equiv \frac{R_1 R_2}{R_2 - R_1}\frac{1}{n-1} = \frac{|R_1||R_2|}{|R_1|+|R_2|}\frac{1}{n-1}$
(R_2 DEFINED AS NEGATIVE)

OUR INPUT WAVE IS

$$U(x,y,z) = A e^{iK(z\cos\theta + x\sin\theta)} \approx A e^{iK(z+\theta x)}$$

THIS GIVES, AT $z=0$,

$$f_0(x) \equiv U(x,y,z=0) = A e^{iK\theta x} = A e^{i2\pi\frac{\theta x}{\lambda f}}$$

NOW WE WANT TO PROPOGATE THE WAVE TO f_1.

$$\tilde{f}_0(\nu_x,\nu_y) = A\int dx\,dy\, f_0(x,y)\, e^{-i2\pi(\nu_x x+\nu_y y)}$$

$$= A\int dx\,dy\, f_0(x,y)\, e^{-i2\pi[\nu_y y + (-\frac{K\theta}{2\pi}+\nu_x)x]}$$

$$= A\,\delta(\nu_y)\,\delta(\nu_x - \tfrac{K\theta}{2\pi}) \quad \text{AND } \tfrac{K}{2\pi} = \tfrac{1}{\lambda}$$

$$= A\,\delta(\nu_y)\,\delta(\nu_x - \tfrac{\theta}{\lambda})$$

$$f_1(x,y) = \int d\nu_x\,d\nu_y\, \tilde{f}_0(\nu_x,\nu_y)\, e^{i2\pi(\nu_x x+\nu_y y)}\, e^{iK_z f}\Big|_{K_z = \sqrt{K^2-K_x^2-K_y^2}}$$

$$= A e^{i2\pi \frac{\theta}{\lambda} x} e^{i 2\pi \frac{f}{\lambda}\left[1-\theta^2\right]^{\frac{1}{2}}}$$

$$\approx A e^{i 2\pi \frac{x\theta}{\lambda}} \underbrace{e^{i2\pi \frac{f}{\lambda}} e^{-i 2\pi \frac{f\theta^2}{2\lambda}}}_{\text{ADDITIONAL PHASE FACTOR.}}$$

WE GET THE SAME WAVE WITH A PHASE FACTOR DUE TO MOVING. NEXT WE APPLY THE LENS

$$t(x,y) f_1(x,y) = h_0 A e^{ikf} e^{-ik\frac{\theta^2}{2}f} \exp\left[-\frac{ik}{2f}(x^2+y^2-2\theta f x)\right]$$

$$= h_0 A e^{ikf} \underbrace{e^{-ik\frac{\theta^2}{2}f} e^{ik\frac{\theta^2}{2}f}}_{=1} \exp\left[-\frac{ik}{2f}((x-\theta f)^2+y^2)\right]$$

$$= h_0 A e^{ikf} e^{-ik\frac{(x-\theta f)^2+y^2}{2f}}$$

NOW THIS IS A PARABOLODIAL WAVE CENTERED AT $(x,y,z) = (\theta f, 0, f)$. AS WE SHOWED IN THE PREVIOUS PROBLEM SET, THE LENS CONVERTED A PLANE WAVE INTO A PARABOLODIAL WAVE. NEXT WE NEED TO PROPOGATE THIS WAVE, SO WE USE THE FRESNEL APPROXIMATION.

$$f_2(x,y) = \frac{-ie^{ikf}}{\lambda f}\int dx'dy' \underbrace{t(x',y') f_1(x',y')}_{\text{WAVE AFTER LENS.}} e^{i\pi\frac{(x-x')^2+(y-y')^2}{\lambda f}}$$

$$= \frac{-i}{\lambda f} h_0 A e^{ikf} \underbrace{\int dx' e^{ik\frac{(x-x')^2-(x'-\theta f)^2}{2f}}}\int dy' e^{ik\frac{(y-y')^2-(y')^2}{2f}}$$

$$\int dx' e^{ik\frac{(x-x')^2-(x'-\theta f)^2}{2f}} = \int dx' \exp\left[\frac{ik}{2f}(x^2-2xx'+2x'\theta f-\theta^2 f^2)\right]$$

$$= e^{-ik\frac{\theta^2}{2}f} e^{ik\frac{x^2}{2f}} \underbrace{\int dx' \exp\left[-i\frac{kx'}{f}(x-\theta f)\right]}_{\lambda f \delta(x-\theta f)}$$

$$= e^{ik\frac{x^2}{2f}} e^{-ik\frac{\theta^2}{2}f} \lambda f \delta(x-\theta f)$$

LIKEWISE

$$\int dy' \exp\left[\frac{ik}{2f}((y-y')^2-y'^2)\right] = \lambda f e^{ik\frac{y^2}{2f}} \delta(y)$$

So We Have

$$f_2(x,y) = (\lambda f)^2\left(\frac{-i}{\lambda f}\right) h_0 A\, e^{i2kf}\, e^{-ik\frac{\theta^2}{2}f}\, e^{ik\frac{x^2+y^2}{2f}}\, \delta(x-\theta f)\,\delta(y)$$

$$= -i\lambda f\, h_0 A\, e^{i2kf}\, e^{-ik\frac{\theta^2}{2}f}\, e^{ik\frac{x^2+y^2}{2f}}\, \underbrace{\delta(x-\theta f)\,\delta(y)}_{\text{This is a point at } (x,y)=(\theta f, y)}$$

We Have The Fourier Transform Of Our Initial Wave (A Delta Spike At $(\theta f, y)$ In The Mask Plane).

Next We Propogate In The Fresnel Approximation Again.

$$f_3(x,y) = \frac{-i}{\lambda f} e^{ikf} \int dx'dy'\, f_2(x',y')\, e^{ik\frac{(x-x')^2+(y-y')^2}{2f}}$$

$$= -h_0 A e^{i3kf}\, \underbrace{e^{-ik\frac{\theta^2}{2}f}\, e^{ik\frac{\theta^2}{2}f}}_{=1}\, e^{ik\frac{(x-\theta f)^2+y^2}{2f}}$$

$$= -h_0 A e^{i3kf}\, e^{ik\frac{(x-\theta f)^2+y^2}{2f}}$$

Again We Have A Paraboloidal Wave, Which We Need To Pass Through The Other Lens.

$$t(x,y) f_3(x,y) = -A h_0^2\, e^{-ik\frac{x^2+y^2}{2f}}\, e^{ik\frac{(x-\theta f)^2+y^2}{2f}}$$

$$= -A h_0^2\, e^{i3kf}\, e^{ik\frac{\theta^2}{2}f}\, e^{-ik\theta x}$$

We Then Fourier Transform In Order To Propogate Through Free Space

$$\tilde{f}(\nu_x,\nu_y) = \int dx\,dy\; t(x,y) f_3(x,y)\, e^{-i2\pi(\nu_x x+\nu_y y)}$$

$$= -A h_0^2\, e^{i3kf}\, e^{ik\frac{\theta^2}{2}f} \int dx\,dy\, e^{-ik\theta x}\, e^{-i2\pi(\nu_x x+\nu_y y)}$$

$$= -A h_0^2\, e^{i3kf}\, e^{ik\frac{\theta^2}{2}f} \underbrace{\int dx\, e^{-i2\pi\left(\frac{\theta}{\lambda}+\nu_x\right)x}}_{\delta\left(\nu_x+\frac{\theta}{\lambda}\right)} \underbrace{\int dy\, e^{-i2\pi\nu_y y}}_{\delta(\nu_y)}$$

$$= -A h_o^2 e^{i3Kf} e^{i2K\frac{\theta^2}{2}f} \delta(\nu_x + \tfrac{\theta}{\lambda}) \delta(\nu_y)$$

AND FINALLY

$$f_4(x,y) = \int d\nu_x d\nu_y \, \tilde{f}(\nu_x, \nu_y) e^{i2\pi(\nu_x x + \nu_y y)} e^{iK_z f} \Big|_{K_z = \sqrt{K^2 - K_x^2 - K_y^2}}$$

$$= -A h_o^2 e^{i3Kf} e^{iK\frac{\theta^2}{2}f} e^{-iK\theta x} e^{iK(1-\theta^2)^{\frac{1}{2}}f}$$

$$= -A h_o^2 e^{i4Kf} e^{-iK\theta x}$$

$$= -h_o^2 e^{i4Kf} f_o(-x)$$

SO WE GET $f_4(x) \propto -f_o(-x)$ WITH A PHASE FACTOR $h_o^2 e^{i4Kf}$. THE h_o^2 COMES FROM THE TWO LENSES AND THE e^{i4Kf} COMES FROM TRAVELING $4f$ DISTANCE THROUGH THE APPERATUS.