PHYS 408 Group Activities for Mar 10 2011

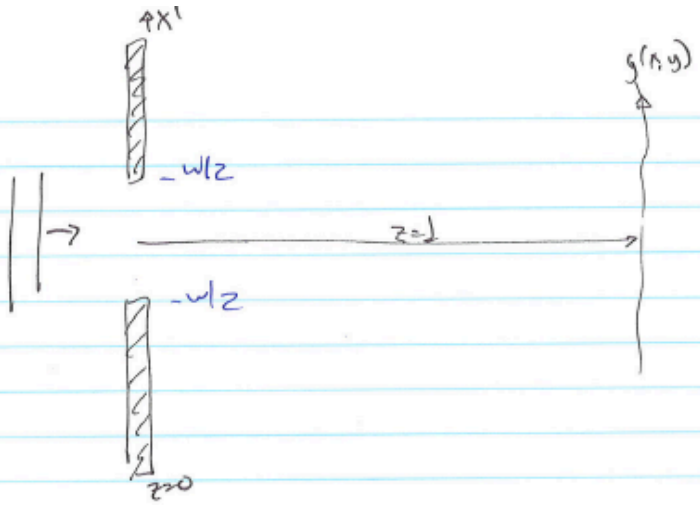


1- Consider a plane wave traveling in the +z direction that hits a square aperture in x-z plane and lets apply the Huygens principle to see what the resulting diffraction pattern is. For these exercises there is no need to evaluate the expressions/integrals

a) Consider a point source (a spherical wavelet) at (x',z=0). What is the result at z=d?
b) Now sum up all possible wavelets at z=0 to get an expression for the image g(x) at z=d.
c) Explain in your own words how you have just used the Huygens principle

2- Repeat 1b for a square aperture illuminated by a plane wave traveling at an angle θ with respect to the z-axis

Bonus: If there is a transparency in the aperture with transmittance $t(x)=\exp(i2\pi x/\Lambda)$ repeat 1b



1 a) for a point source located @ $x'=0$, $z=0$

@ $z=0$ $\quad f(x') = \delta(x')$

@ $z=d$ $\quad g(x) = \frac{A}{d} e^{ik\sqrt{x^2+d^2}} \simeq \frac{A}{d} e^{ikd\left(1+\frac{x^2}{2d^2}\right)}$

but problems asked for an arbitrary position for δ impulse

$\Rightarrow \quad f(x') = \delta(x-x')$

so $g(x) = \frac{A}{d} e^{ik\left(1+\frac{(x-x')^2}{2d^2}\right)}$ for a point source @ an arbitrary x' @ $z=0$

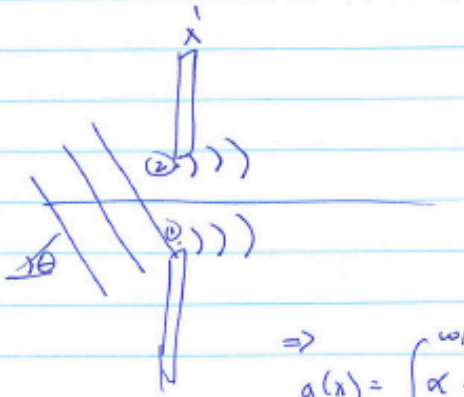
b) Sum up over all none zero points in x' plane:

$$g(x) \propto \int_{-w/2}^{w/2} \frac{A}{d} e^{ikd\left(1+\frac{x^2}{2d^2}\right)} dx$$

$$g(x) = \int_{-w/2}^{w/2} \alpha \frac{A}{d} e^{ikd\left(1+\frac{(x-x')^2}{2d}\right)} dx'$$ where α is a constant

as pointed out by Hardeep you need to add that constant α to make the units work out. Don't worry too much about this point: it is a detail... but not the most important

c) by summing over spherical wavelets over all non-zero wavefronts in the input plane, you can find the output @ $z=d$ (which is Huygen's principle)



Now point source (1) has a phase delay relative to (2)

That phase delay is $k\sin\theta x'$ (doesn't depend on x)

$$\Rightarrow g(x) = \int_{-w/2}^{w/2} \alpha \frac{A}{d} e^{ikd\left(1+\frac{(x-x')^2}{2d^2}\right)} e^{ik\sin\theta x'} dx' \quad (1)$$

as it turns out $\alpha = \frac{i}{\lambda}$

comparing (1) w/ $g(x) = \int_{-\infty}^{\infty} f(x') h(x-x') dx'$

- $f(x')$ → amplitude/phase of spherical wavelet.
- $h(x-x')$ → impulse response of free space ("The result of an input impulse")

so by just considering physics you have derived the real space method to finding the output image => you convolve the input w/ the impulse response:

PHYS 408 Group Activities for Mar 10 2011



You derived from physics principles a different way to calculate the output image g(x,y).

$$g(x,y) = \frac{i}{\lambda d}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' f(x',y') \exp\left[ikd(1 + \frac{(x-x')^2 + (y-y')^2}{2d^2}\right]$$

Amplitude/phase of point source or spherical wavelet

Impulse response function h(x,y) of free space: Fresnel approx of spherical wave at (x,y, z=d) from point source, impulse at (x',y', z=0)

$$g(x,y) = f(x,y) \otimes h(x,y)$$

We can also use the fourier recipe as before to find g(x,y)

$$\tilde{g}(k_x,k_y) = \tilde{f}(k_x,k_y)\tilde{h}(k_x,k_y)$$

$$g(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(k_x,k_y)\tilde{h}(k_x,k_y)e^{i(k_x x + k_y y} dk_x dk_y$$

Fourier Transform of impulse response function is the transfer function

$$\tilde{h}(k_x, k_y) = \mathcal{F}[h(x, y)]$$

Impulse Response

$$g(x, y) = f(x, y) \otimes h(x, y)$$

$$\tilde{g}(k_x, k_y) = \mathcal{F}[g(x, y)] = \mathcal{F}[f(x, y) \otimes h(x, y)]$$

Fourier Recipe

$$\tilde{g}(k_x, k_y) = \tilde{f}(k_x, k_y)\tilde{h}(k_x, k_y)$$

$$\tilde{g}(k_x, k_y) = \mathcal{F}[f(x, y)]\mathcal{F}[h(x, y)]$$

Convolution Theorem

$$\mathcal{F}[f(x, y) \otimes h(x, y)] = \mathcal{F}[f(x, y)]\mathcal{F}[h(x, y)]$$

$$\mathcal{F}[f(x, y)h(x, y)] = \mathcal{F}[f(x, y)] \otimes \mathcal{F}[h(x, y)]$$

PHYS 408 Group Activities for Mar 10 2011

Please complete these exercises without opening the textbook. If you have questions talk with your group mates or raise your hand and David will come to help

$$g(x,y) = f(x,y) \otimes \; h\,(x,y) \; = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy' f(x',y')h(x-x',y-y')$$

3- Complete the convolution to find g(x) and sketch the result for the following cases (just ignore the y dimension for simplicity…*hint remember from your reading how to visualize convolution: there is no need to actually do the integration*

a) $f(x) = rect\left[\frac{x}{W}\right] \quad h(x) = \delta(x)$

b) $f(x) = rect\left[\frac{x}{W}\right] \quad h(x) = \delta(x-x_o) + \delta(x+x_o) \quad x_o > W$

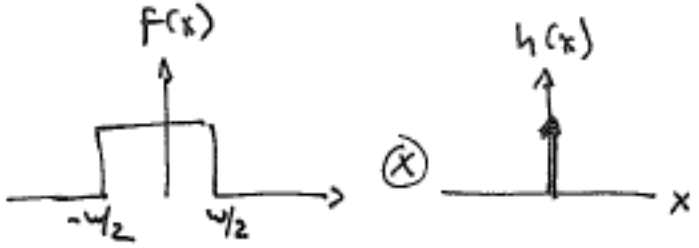
c) $f(x) = rect\left[\frac{x}{W}\right] \quad h(x) = rect\left[\frac{x}{W}\right]$

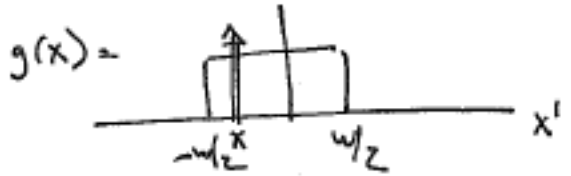
Bonus: Use the convolution theorem to find $\mathcal{F}[g(x)]$ in 1 b)

$$g(x,y) = f(x,y) \otimes g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy' f(x',y')h(x-x',y-y')$$

1 a) $f(x) = \text{rect}\left(\frac{x}{w}\right)$

$h(x) = \delta(x)$





$g(x) = f(x) \otimes h(x)$

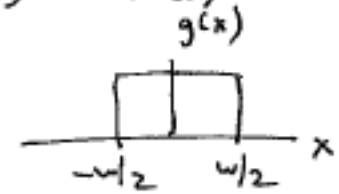
$= \text{rect}\left(\frac{x}{w}\right)$ ← any function convolved w/ $\delta(x)$ gives same function again

to do the math:

$$g(x) = \int_{-\infty}^{\infty} dx' \,\text{rect}\left(\frac{x'}{w}\right)\delta(x-x')$$

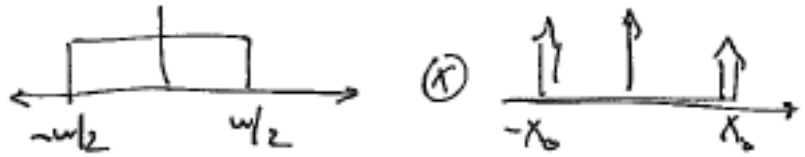
↳ only non-zero w/ $x = x'$

$\therefore g(x) = \text{rect}\left(\frac{x}{w}\right)$

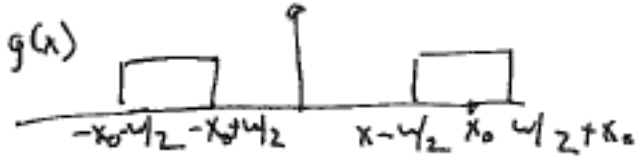


$$g(x,y) = f(x,y) \otimes g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy' f(x',y')h(x-x',y-y')$$

b)



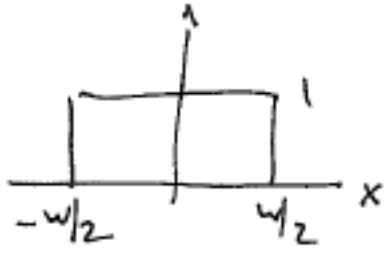
visually flip $h(x) \rightarrow h(x-x')$ & slide Thru $f(x)$

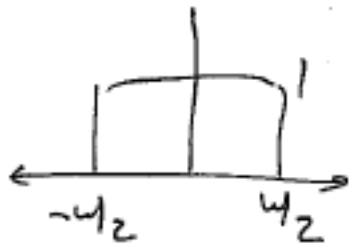


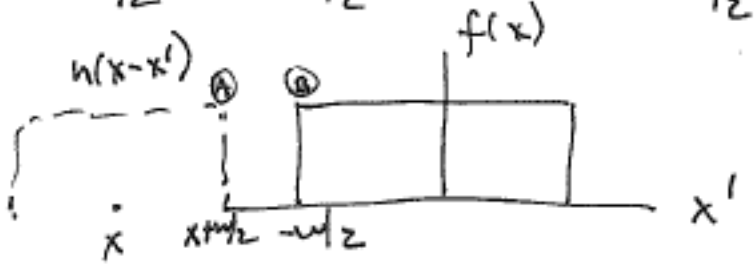
math is very similar to a) so I won't ~~write~~ repeat it.

$$g(x,y) = f(x,y) \otimes g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx'dy' f(x',y') h(x-x', y-y')$$

c)



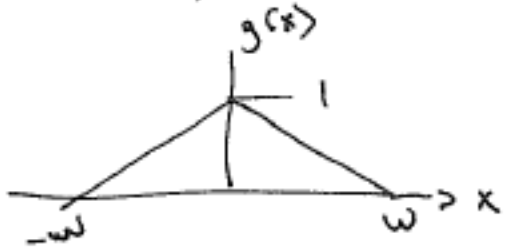




only start to get non-zero overlap when A & B are the same
i.e. $x = -w$

overlap increases linearly until $x=0$ then decreases until $x=w$

=>



c) hence

PHYS 408 Group Activities for Mar 10 2011



You derived from physics principles a different way to calculate the output image g(x,y).

$$g(x,y) = \frac{i}{\lambda d}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' f(x',y') \exp\left[ikd(1+\frac{(x-x')^2+(y-y')^2}{2d^2}\right]$$

Amplitude/phase of point source or spherical wavelet

Impulse response function h(x,y) of free space: Fresnel approx of spherical wave at (x,y, z=d) from point source, impulse at (x',y', z=0)

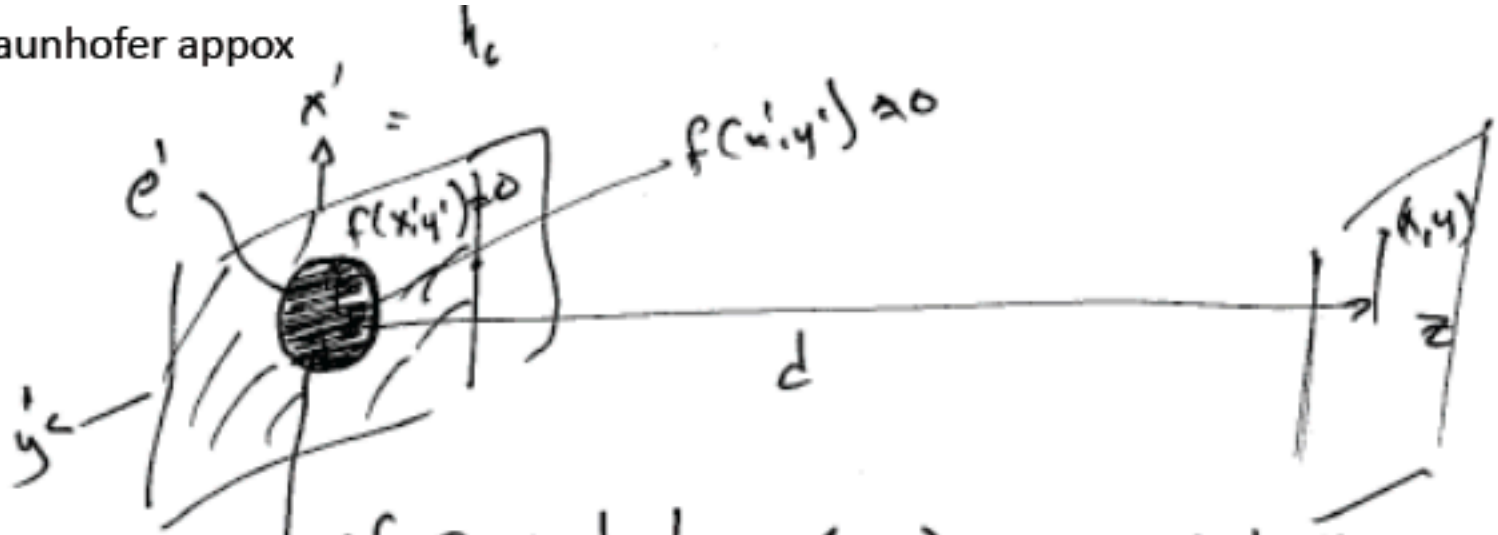
4- Now lets evaluate g(x,y) in the far-field (meaning d is large).

(a) With d being large, see if you can simplify the above integral. Put some limits on d for this simplification and justify your approximation with physics.

(b) Assume an aperture size of 1 cm and a wavelength of 500 nm, at what distance is z=d in the far field?

Bonus: Evaluate g(x,y) for the input f(x,y)=rect(x/L)*rect(y/2L) in the far-field

Fraunhofer appox



if @ input plane ($z=0$) we restrict the input image to a small region w/in a radius ρ' of z axis $\Rightarrow$ $f^{eq}(x',y')$ is non-zero only in

such that $\frac{k}{2d}\rho'^2 \ll \pi$ Then we can neglect $\rho'^2 = x'^2 + y'^2$

this phase term in the integrand

$$g(x,y) \simeq h_0 e^{i\frac{k}{2d}(x^2+y^2)} \iint_{-\infty}^{\infty} f(x',y') e^{-i\frac{k}{d}(xx'+yy')} dx'dy'$$

PHYS 408 Group Activities for Mar 10 2011

From our work on today you derived from physics principles a different way to calculate the output image g(x,y). In doing so, you "derived" the Famous Huygens-Fresnel integral.

$$g(x,y) = \frac{i}{\lambda}\iint_{-\infty}^{\infty} f(x',y')\,\frac{e^{ikz}e^{i\frac{k}{2z}((x-x')^2+(y-y')^2)}}{z}\,dx'dy'$$

Amplitude/phase of point source or spherical wavelet

Impulse response function h(x,y) of free space: Fresnel approx of spherical wave at (x,y, z=d) from point source, impulse at (x',y', z=0)

Then, we GREATLY simplified this expression by assuming we are in the far field $k\frac{x'^2+y'^2}{2d} \ll \pi$ (i.e. $exp\left\{i\frac{k}{2z}(x'^2+y'^2)\right\} \to 1$). The Fraunhofer approximation is also famous in its own right.

$$g(x,y) = \frac{i}{\lambda z}e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}\iint_{-\infty}^{\infty} f(x',y')e^{-i\frac{k}{z}[xx'+yy']}\,dx'dy'$$

Phase term in the observation plane (x,y)
Unimportant when calculating intensity pattern

Impulse response function h(x,y) of free space are now PLANE WAVES!!! Instead of paraboloidal waves.

5- From the above expression, show that:

$$g(x,y) = h_0 e^{i\frac{k}{2z}(x^2+y^2)}\mathcal{F}\{f(x',y')\}_{\substack{k_x=kx/z \\ k_y=ky/z}}$$

and then explain in words what the far-field image is in terms of the input image.