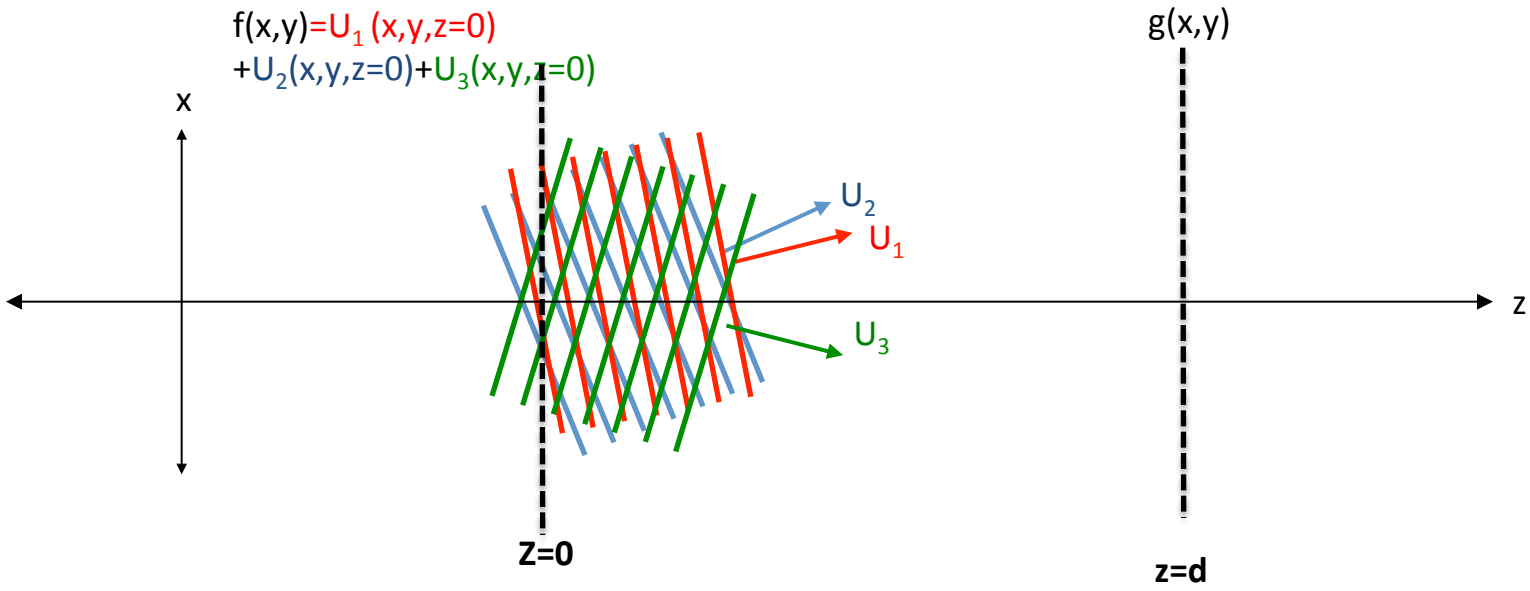
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Please complete these exercises without opening the textbook. If you have questions talk with your group mates or raise your hand and David or Ellen will come to help

Review last activity from last week

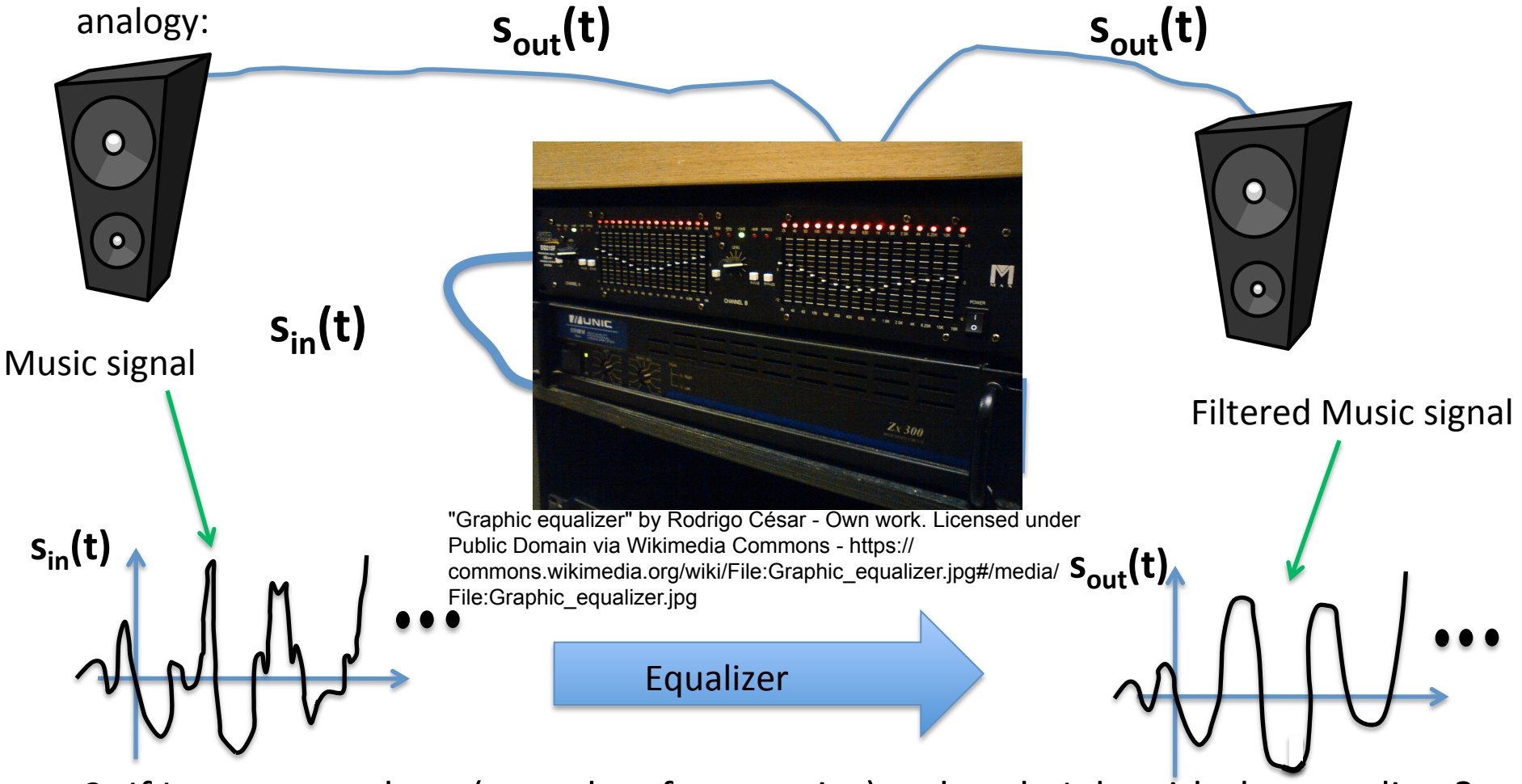


1- The field U(x,y,z) at z=0 can be decomposed into 3 plane waves (U_1,U_2, and U_3) at evaluated at z=0. In other words, the input image f(x,y)=U(x,y,z=0) and can be constructed by the interference of 3 plane waves as shown. At z=d find the (new) output image g(x,y)=U(x,y,z=d).

2- In your own words explain what you just did mathematically to get the output image g(x,y).

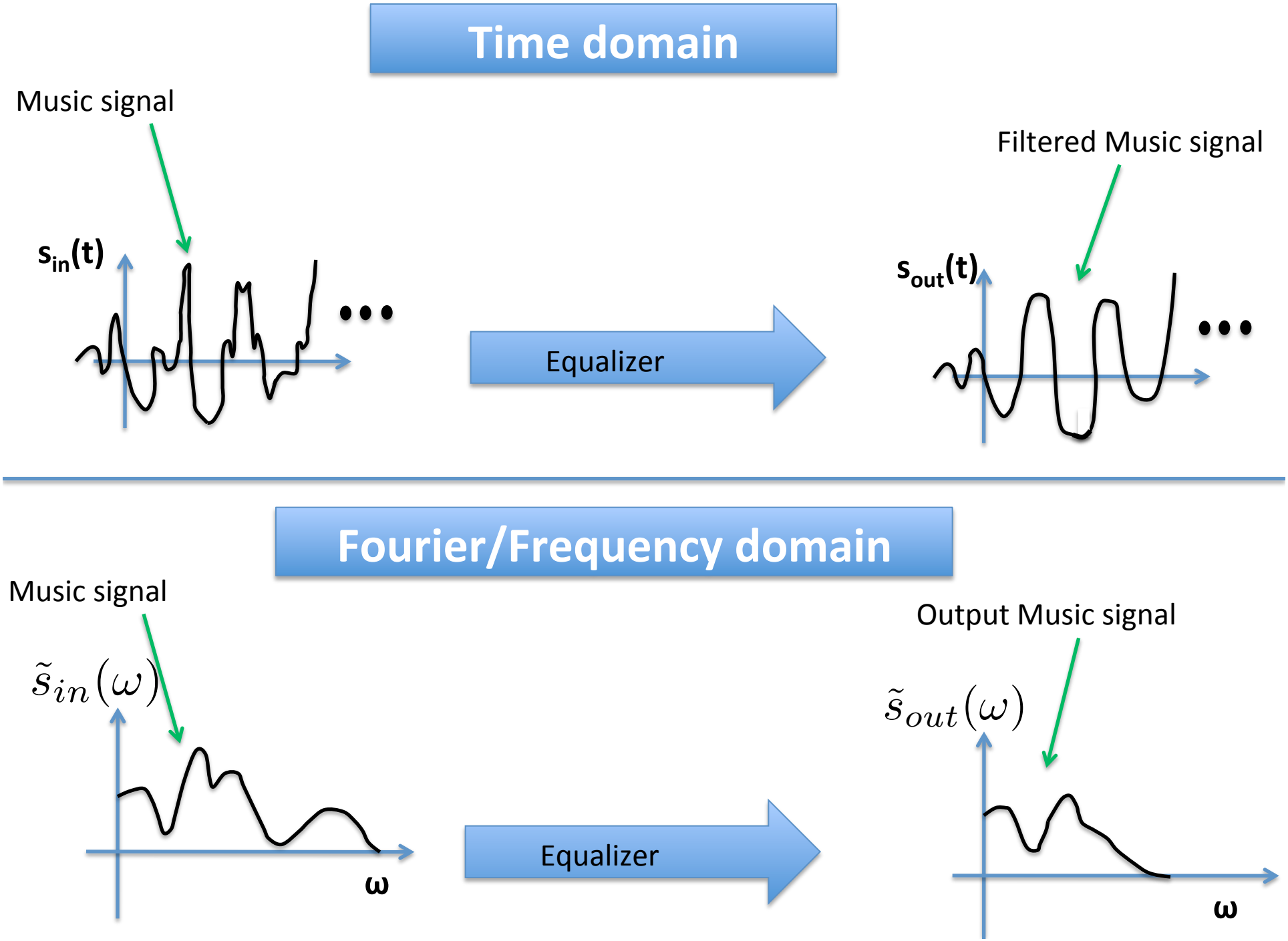
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There is a better set of tools to "propagate" and sum all of the plane waves from z=0 to z=d to find the new output image. To understand the physics of these tools consider this analogy:

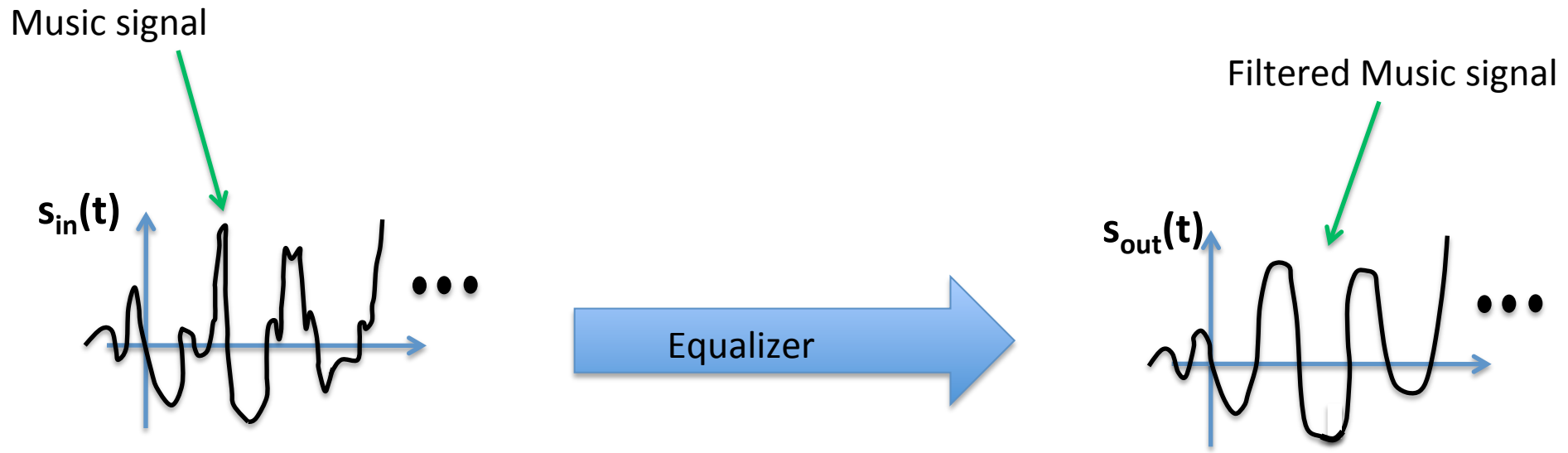


"Graphic equalizer" by Rodrigo César - Own work. Licensed under Public Domain via Wikimedia Commons - https://commons.wikimedia.org/wiki/File:Graphic_equalizer.jpg#/media/File:Graphic_equalizer.jpg

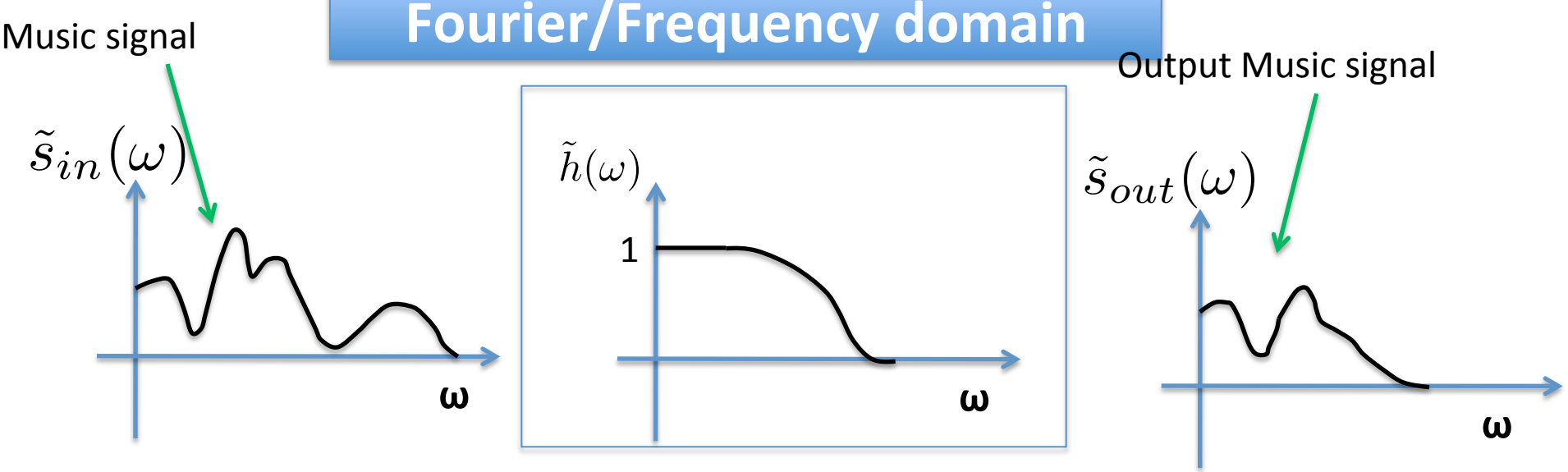
3- If I want more bass (more low frequencies)…what do I do with the equalizer? What does the equalizer do to input signal? How could you describe in several sentences its effect mathematically (Hint: think beyond the time domain).


Time domain
Music signal
$s_{in}(t)$
Equalizer
Filtered Music signal
$s_{out}(t)$
Fourier/Frequency domain
Music signal
$\tilde{s}_{in}(\omega)$
ω
Equalizer
Output Music signal
$\tilde{s}_{out}(\omega)$
ω

Time domain



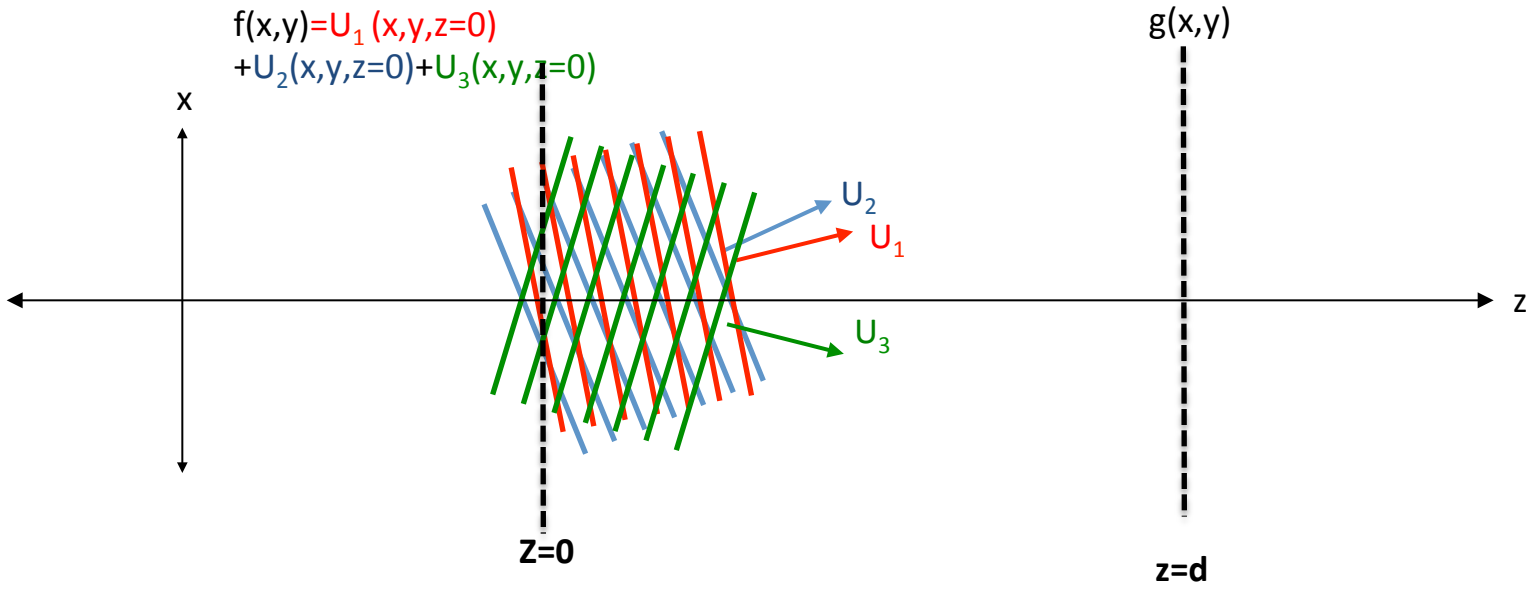
Fourier/Frequency domain



Each individual frequency in $S_{in}(\omega)$ is multiplied by the corresponding value in $H(\omega)$...yielding $S_{out}(\omega)$

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4-Back to Optics and to complete the analogy…

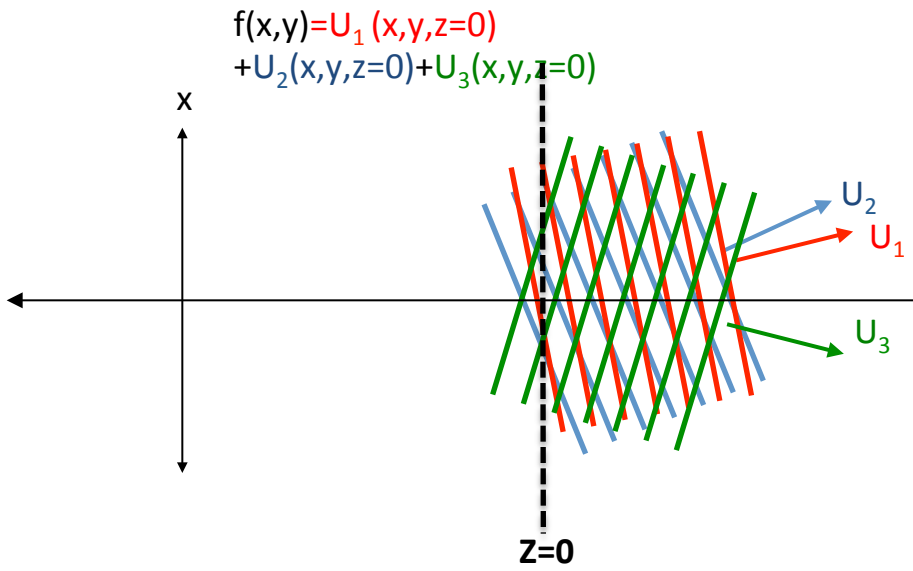
a) What do $s_{in}(t)$ $s_{out}(t)$ $\tilde{s}_{in}(\omega)$ $\tilde{s}_{out}(\omega)$ correspond to in above diagram?

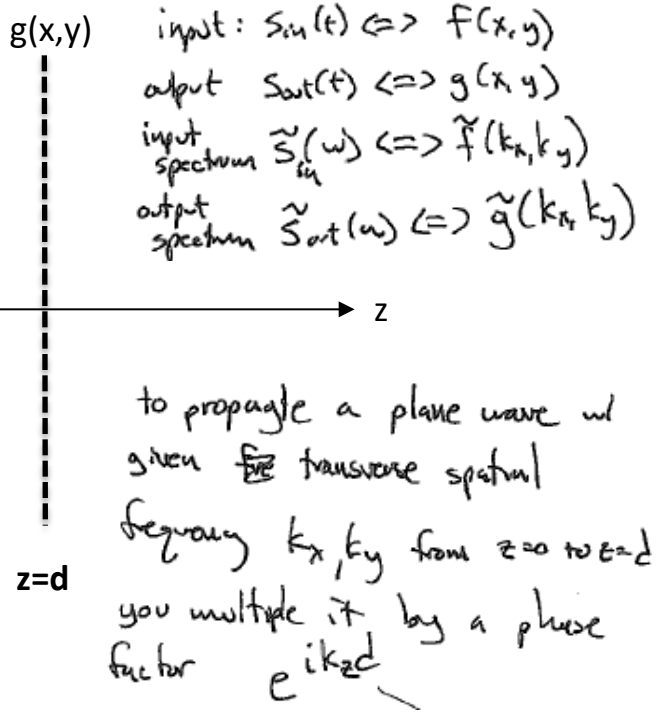
b) Give a mathematical expression for the optical equivalent of free space propagation for $\tilde{h}(\omega)$

c) Explain in words the optical transfer function of free space does to each plane wave

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4-Back to Optics and to complete the analogy…

a) What do **$s_{in}(t)$** **$s_{out}(t)$** $\tilde{s}_{in}(\omega)$ $\tilde{s}_{out}(\omega)$ correspond to in above diagram?

b) Give a mathematical expression for the optical equivalent of $\tilde{h}(\omega)$

music optics

$\therefore$ transfer function $\tilde{h}(\omega) \Leftrightarrow \tilde{h}(k_x,k_y)$

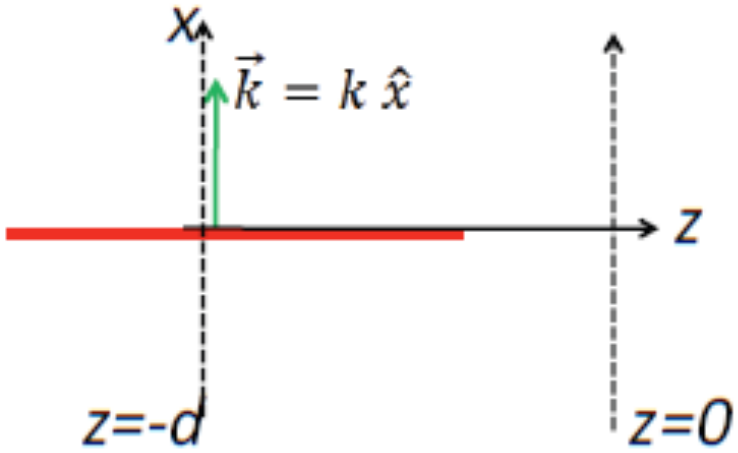
$\therefore \tilde{h}(k_x,k_y) = e^{ikzd} = e^{i\sqrt{k-k_x^2-k_y^2}\,d}$

What is the value of the transfer function $H(k_x, k_y)$ that describes free space propagation from **z=-d** to **z=0** for the plane wave shown below ?
In other words, what is $H(k_x, k_y)$ when $k_x = k, k_y = 0$?

(A) $H(k_x = k, k_y = 0) = 0$

(B) $H(k_x = k, k_y = 0) = -1$

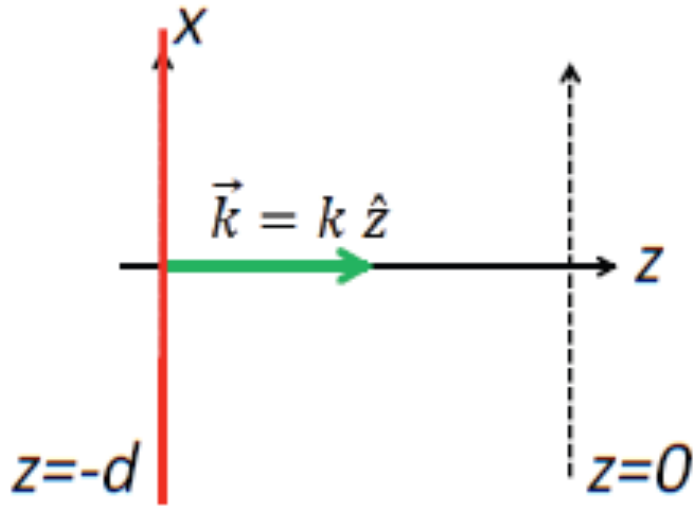
(C) $H(k_x = k, k_y = 0) = 1$



(D) $H(k_x = k, k_y = 0)$ is undefined for plane waves at exactly 90 degrees

(E) $H(k_x = k, k_y = 0) = e^{i\frac{\pi}{2}}$

What is the value of the transfer function $H(k_x, k_y)$ that describes free space propagation from **z=-d** to **z=0** for the plane wave shown below ?
In other words, what is $H(k_x, k_y)$ when $k_x = k_y = 0$?



(A) $H(k_x = 0, k_y = 0) = 1$

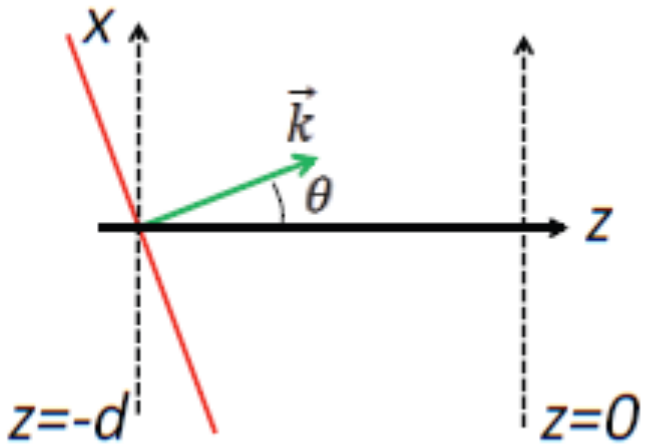
(B) $H(k_x = 0, k_y = 0) = e^{-ikz}$

(C) $H(k_x = 0, k_y = 0) = e^{-ikd}$

(D) $H(k_x = 0, k_y = 0) = e^{ikz}$

(E) $H(k_x = 0, k_y = 0) = e^{ikd}$

What is the value of the transfer function $H(k_x, k_y)$ that describes free space propagation from **z=-d** to **z=0** for the plane wave shown below, where $\vec{k} = k\cos\theta\,\hat{z} + k\sin\theta\,\hat{x}$?



(A) $H(k_x, k_y) = e^{ikd}$

(B) $H(k_x = k\sin\theta, k_y = 0) = e^{-ikd\cos\theta}$

(C) $H(k_x = k\sin\theta, k_y = 0) = e^{ikd\cos\theta}$

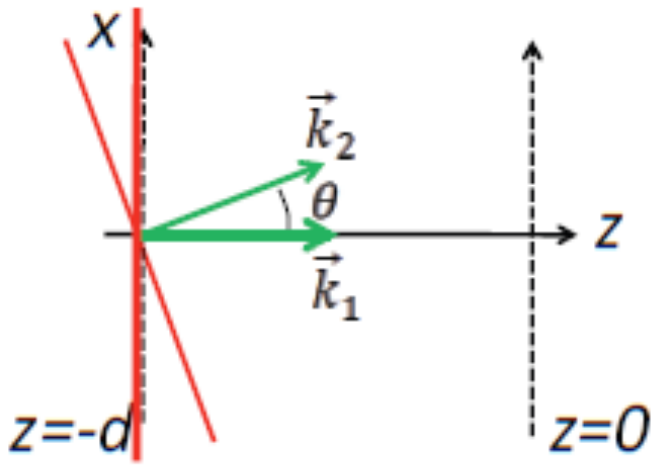
(D) $H(k_x = k\sin\theta, k_y = 0) = e^{ikz\cos\theta}$

(E) $H(k_x = k\sin\theta, k_y = 0) = e^{-ikz\cos\theta}$

What is the transfer function $H(k_x, k_y)$ that describes free space propagation from **z=-d** to **z=0** for the input field: $U(x, y, z = 0) = A_1 e^{i\vec{k}_1 \cdot \vec{r}} + A_2 e^{i\vec{k}_2 \cdot \vec{r}}$

Both beams have the same wavelength and

$\vec{k}_1 = k\,\hat{z}$, $\vec{k}_2 = k\cos\theta\,\hat{z} + k\sin\theta\,\hat{x}$



(A) $H(k_x, k_y) = e^{ikd}$

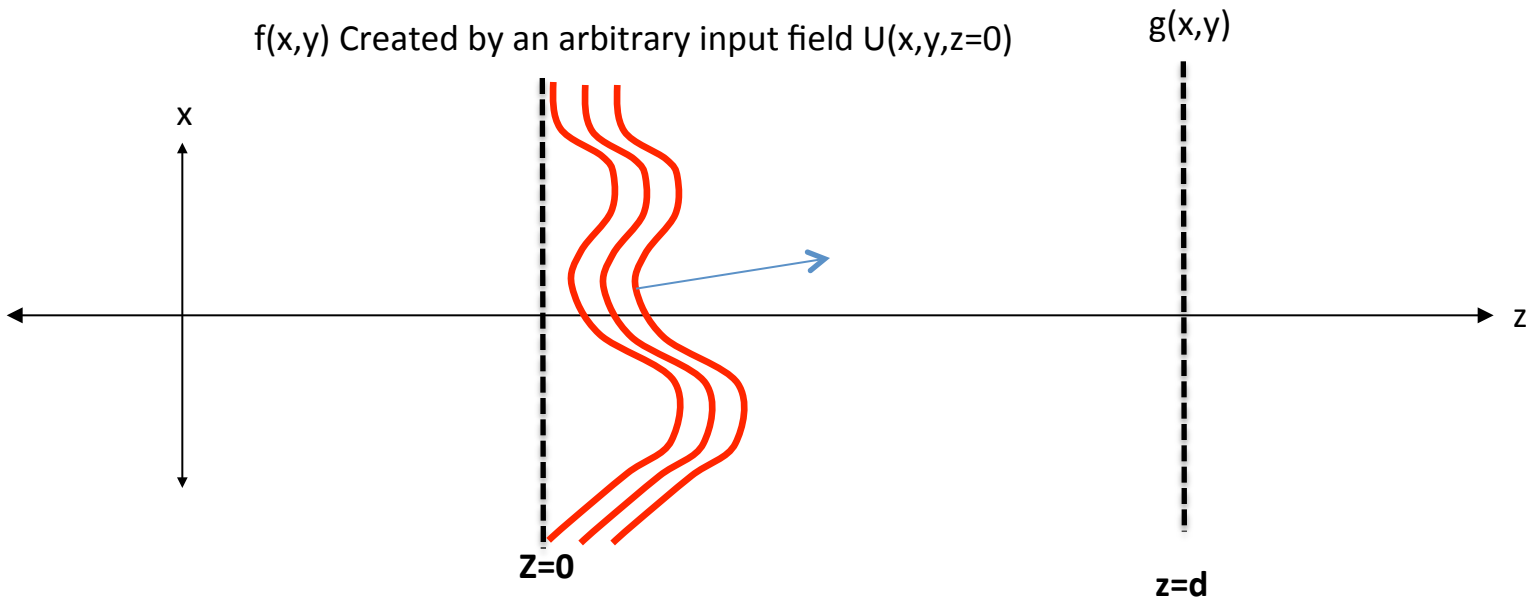
(B) $H(k_x, k_y) = e^{-ikd}$

(C) $H(k_x, k_y) = e^{-ik_z d}$

(D) $H(k_x, k_y) = e^{ik_z d}$

(E) No single expression for ***H*** when having two <u>different</u> plane waves

Prereading Quiz for Today



5- Write down in words the **step by step** Fourier recipe to propagate an arbitrary field f(x,y) in free space from z=0 to z=d. (hint: there should be at least 3 distinct steps, with equations)

Recipe for Fourier Optics

① Math: Evaluate $u(x,y,z=0) = f(x,y)$ ⟷ Physics: get input image $f(x,y)$ can be complex!

② $\tilde{f}(k_x,k_y) = \mathcal{F}[f(x,y)]$ take Fourier Transform

- decompose $f(x,y)$ into a sum of plane waves each propagating @ a particular angle
  - $\theta_x = \frac{k_x}{k_z}$
  - $\theta_y = \frac{k_y}{k_z}$
- corresponding to a spatial frequency k_x, k_y
- which sum @ $z=0$ to create $f(x,y)$

$$\tilde{f}(k_x,k_y) = \iint_{-\infty}^{\infty} f(x,y)e^{-i(k_x x + k_y y)}dxdy$$

③ $\tilde{f}(k_x,k_y)\tilde{h}(k_x,k_y) = \tilde{g}(k_x,k_y)$ ($\tilde{h}$: transfer function)

- Analyze the changes imparted by system on our collection of plane wave

$$\tilde{h}_{Free\ space}(k_x,k_y) = e^{ik_z d} = e^{ik\sqrt{1-\left(\frac{k_x^2+k_y^2}{k^2}\right)}d}$$

④ $g(x,y) = \mathcal{F}^{-1}[\tilde{g}(k_x,k_y)]$

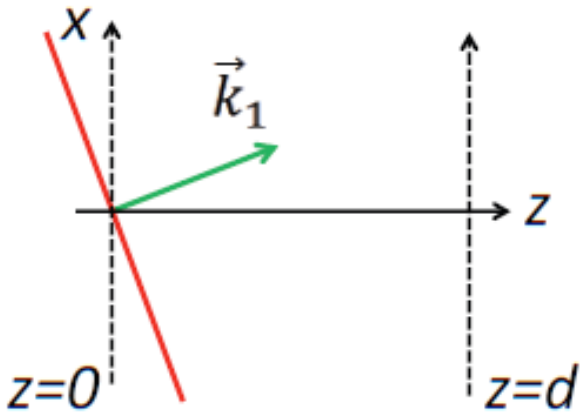
Sum up all plane waves @ $z=d$ to get output image $g(x,y)$

$$g(x,y) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \tilde{g}(k_x,k_y)e^{i(k_x x + k_y y)}dk_x dk_y$$

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A monochromatic plane wave $U(x, y, z) = e^{i(\alpha/2x+\alpha/3y+k_z z)}$ is propagating thru free space from z=0 to z=d, as shown below. (α is a constant). Find g(x,y) at z=d via 2 methods (one method is Fourier Optics).



Bonus: Do the same Fourier method for the following input image $f(x,y) = cos\left(\frac{\alpha}{2}x + \frac{\alpha}{3}y\right) + 1$

step 1 $f(x,y) = e^{i(\frac{\alpha}{2}x + \frac{\alpha}{3}y)}$

step 2 $\tilde{f}(k_x,k_y) = (2\pi)^2 \delta(k_x - \frac{\alpha}{2}) \delta(k_y - \frac{\alpha}{3})$

step 3 $\tilde{g}(k_x,k_y) = e^{ik_z d}(2\pi)^2 \delta(k_x - \frac{\alpha}{2}) \delta(k_y - \frac{\alpha}{3})$

$= e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, d}$

step 4

$$g(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2\pi)^2 \delta(k_x - \frac{\alpha}{2}, k_y - \frac{\alpha}{3}) e^{i(k_x x + k_y y)} e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, d} dk_x dk_y$$

$=$

$=$

$$= e^{i\frac{\alpha}{2}x} e^{i\frac{\alpha}{3}y} e^{i\sqrt{k^2 - \left(\frac{\alpha}{2}\right)^2 + \left(\frac{\alpha}{3}\right)^2}\, d}$$

$$= e^{i\left(\frac{\alpha}{2}x + \frac{\alpha}{3}y\right)} e^{ik_z d}$$

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$$g(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(k_x,k_y)e^{i(k_x x+k_y y)}e^{i\sqrt{k^2-k_x^2-k_y^2}d}dk_x dk_y$$

In the previous example the only reason we could do the integral was because of the delta functions for $\tilde{f}(k_x,k_y)$.

6- Simplify this expression by making the "Fresnel" approximation on the transfer function of free-space. When is this approximation valid in a mathematical and physical sense?

$$\tilde{h}(k_x, k_y) = e^{ikd}$$ ← exact transfer function

$$= e^{ikd\sqrt{1 - \frac{k_x^2 + k_y^2}{k^2}}}$$

→ expand in a Taylor series

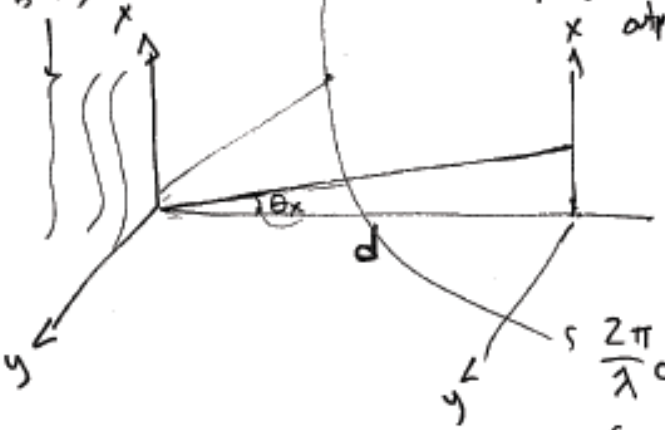
$$= e^{ikd\left[1 - \frac{k_x^2+k_y^2}{2k^2} + \frac{1}{8}\left(\frac{k_x^2+k_y^2}{k^2}\right)^2 \cdots\right]}$$

$$\simeq e^{ikd\left[1 - \frac{k_x^2+k_y^2}{2k^2}\right]}$$ ← Fresnel Approx to free space

if $\frac{1}{8}\left(\frac{k_x^2+k_y^2}{k^2}\right)^2 \ll 1$

if $kd\,\frac{1}{8}\left[\frac{k_x^2+k_y^2}{k^2}\right]^2 \ll \pi$ does n't contribute much to phase

$f(x,y)$ what is this condition physical



output plane

$\theta_x \leftrightarrow k_x$

$\theta_y \leftrightarrow k_y$

$\theta_x = \frac{k_x}{k}$

$\theta_y = \frac{k_y}{k}$

$\frac{2\pi}{\lambda} d \frac{1}{8}\left(\theta_x^2 + \theta_y^2\right)^2 \ll \pi$

$\frac{(\theta_x^2+\theta_y^2)^2}{4} \ll \frac{\lambda}{d}$