→ Homework #8 posted tonight, will be due April 1st (Wedn) @ 5 PM

→ Today will finish summary of approximations & hence finish Fourier Optics

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Fraunhofer approx: - used w/ free space propagation
- a condition that is more restrictive than Fresnel approx where you are in the far-field

Math

$$\underbrace{g(x,y)}_{\text{output image}} = \underbrace{f(x,y)}_{\text{input image}} \circledast \underbrace{h(x,y)}_{\text{impulse response of free space}}$$

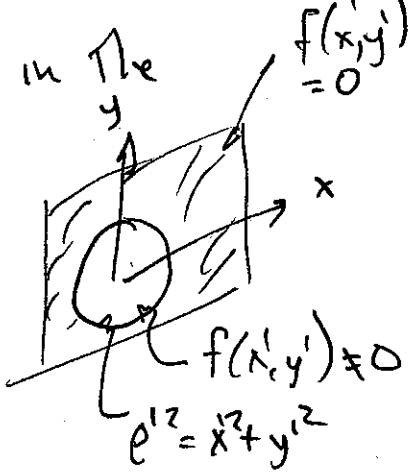
by writing $h(x,y)$ in this form means that you already have made the Fresnel approx

$$= \frac{i}{\lambda d} e^{ikd} \int_{-\infty}^{\infty} f(x',y') \underbrace{e^{i\frac{k}{2d}\left[(x-x')^2+(y-y')^2\right]}}_{h} dx'dy'$$

$$= h_0 e^{i\frac{k}{2d}(x^2+y^2)} \int_{-\infty}^{\infty} f(x',y') e^{i\frac{k}{2d}(x'^2-xx'+y'^2-yy')} dx'dy'$$

start of the Fraunhofer:

(#1) $f(x',y')$ is confined to a small radius in the input plane



$$\frac{k}{2d}(x'^2+y'^2) \ll \pi$$

$$\Rightarrow \frac{k}{2d}(\rho'^2) \ll \pi$$

given ~~result~~ length scale of input image (i.e. where it is non-zero) gives a condition on d.

The second condition for Fraunhofer is in the output plane. if we restrict $g(x,y)$ to a radius ρ

$$\sim 10^{-9} \qquad \frac{\pi}{\lambda d} = \frac{k}{2d}(x^2+y^2) \ll \pi$$

$$\frac{k}{2d}(\rho^2) \ll \pi$$

Then

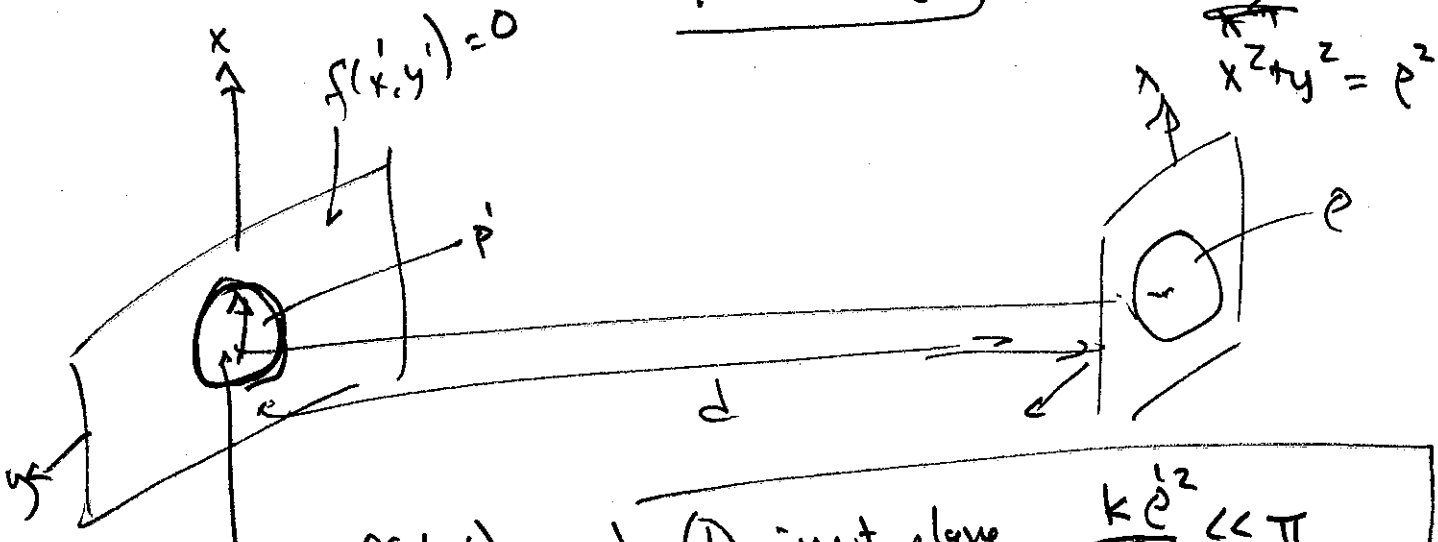
$$g(x,y) \triangleq h_0 \int_{-\infty}^{\infty}\int f(x',y')\, e^{-i\frac{k}{2d}(xx'+yy')}\, dx'dy'$$

$$\boxed{g(x,y) = h_0\, \mathcal{F}(k_x, k_y)} \quad \leftarrow \text{this approx form of } g(x,y) \text{ is only valid within a } \rho \text{ such that}$$

$$k_x = \frac{xk}{d}$$

$$k_y = \frac{yk}{d}$$

$$x^2+y^2 = \rho^2 \ll \frac{\pi 2d}{k}$$



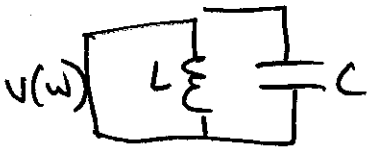
① input plane $\frac{k\rho'^2}{2d} \ll \pi$

② output plane $\frac{k\rho^2}{2d} \ll \pi$

New Topic: Optical Resonators.

in general a resonator traps or stores energy via constructive interference @ certain frequencies

electrical resonator



acoustic

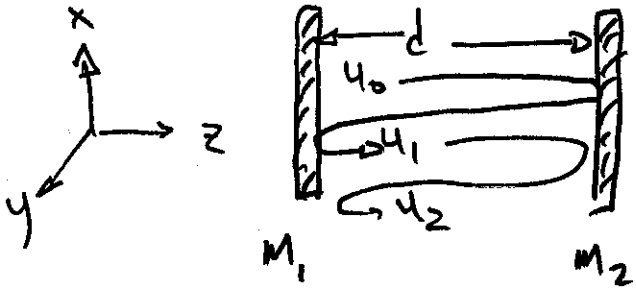
flute, bottle

mechanical resonator

- mass/spring
- quartz crystal
- bridge

so an optical resonator traps or store light only @ certain frequencies (or wavelengths)

The simplest optical resonator is 2 flat mirrors



- $M_{1,2}$ are 100% reflective (perfect)
- a Fabry-Perot resonator or cavity

So to find what frequencies are resonant... must satisfy:

(1) The field must still solve the Helmholtz wave eqn. (Otherwise not an E&M field)

(2) Boundary conditions imposed by mirrors must satisfied.

(3) The field must add constructive in phase
⇒ a phase shift of $2\pi n$ must occur b/w u_0 & u_1; u_1 & u_2

Still have polarization aside for now

assume initial field U_0

→ after 1 roundtrip $U_1 = U_0 e^{i \Phi_{RT}(x,y,z)}$

R.T. phase $\Phi_{RT}(x,y,z) = \underbrace{\Theta_{RT}(x,y)}_{\text{transverse phase}} + \underbrace{\phi(z)}_{\text{longitudinal phase}}$

In the case of planar mirrors, they don't affect the transverse phase

⇒ This ~~motivates~~ motivates us to guess that the "eigen-modes" of the planar ~~resonator~~ resonator are: plane waves ← b/c plane wave has no transverse phase dependence

back to quantum

$\hat{H}|\psi\rangle = E|\psi\rangle$

$\Rightarrow U \sim e^{ikz}$ → represents the transverse field component

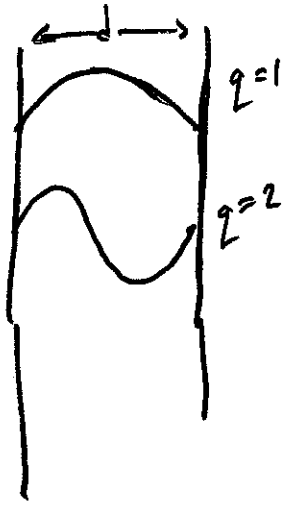
to satisfy the boundary conditions imposed by mirrors

$U(0) = U(d) = 0$ ← b/c transverse E fields must be zero @ metal surface

$U = A_1 e^{ikz} + A_2 e^{-ikz}$

$A_1 = -A_2$ b/c BC ~~⇔~~ conditions to be satisfied @ $z=0$

$\Rightarrow u \cong A \sin kz$; for $u(d)=0$



$\Rightarrow$ gives $|kz| = q\pi$

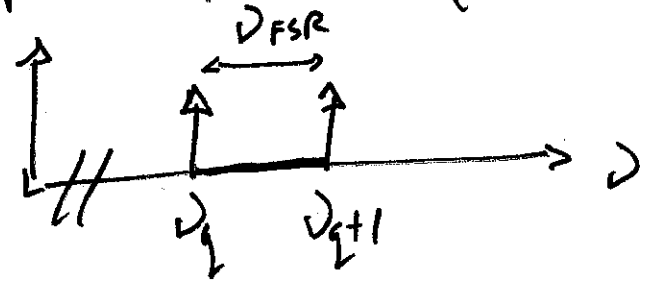
$z = d$

$\Rightarrow \boxed{k_q = \frac{\pi}{d} q}$ — only specific k's are resonate

- or -

$\boxed{\nu_q = \frac{c}{2d} q}$ — only certain frequencies exist

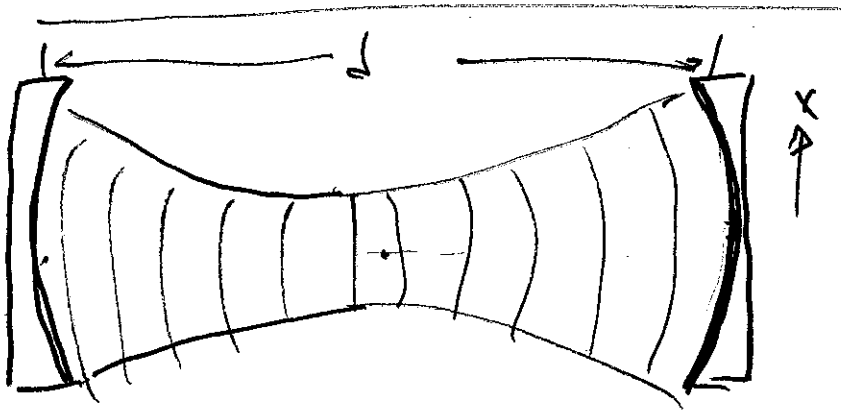
plot the possible frequencies of the field in resonator



free spectral range

$\Rightarrow \nu_{FSR} = \nu_{q+1} - \nu_q$

$\boxed{\nu_{FSR} = \frac{c}{2d}}$



Q: are plane waves resonate in a cavity w/ curved mirrors?

A: No b/c they won't self-consistent in x,y after 1 roundtrip

... but a gaussian beam would be resonant in this confocal resonator... why?

b/c it has a parabolic phase front that can be resonate as long as the phases fronts @ the mirrors match the mirrors' curve

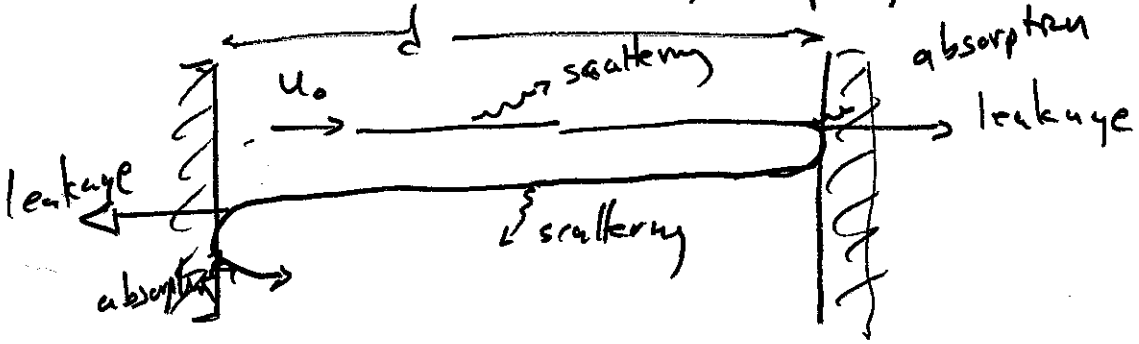
Main point is that you need to consider both transverse & longitudinal phases in determining resonant ~~condition~~ conditions / frequencies

— Back to planar mirrors —

in reality mirrors aren't perfect (not R = 100%)

How does this change our analysis?

lossy ("real life", damped) ~~R~~esonators



Amplitude of the field in RT is reduced by

- absorption on mirrors
- ~~leakage~~ leakage
- scattering by mirrors

} → lump all of these effects in one factor ⇒ "r"

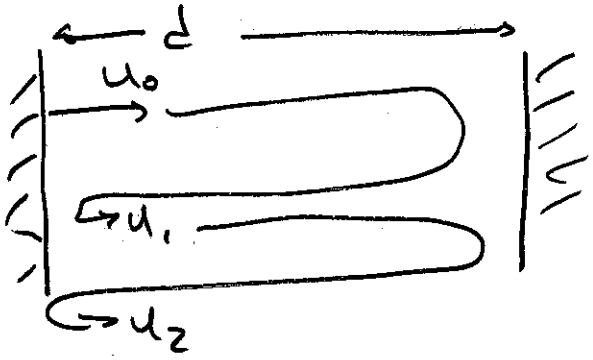
- on each round trip

$$|u_{n+1}| = r\,|u_n| \quad ; \quad r<1 \text{ , real , } r \text{ constant for every round trip}$$

- still have a longitudinal round trip phase

$$\phi = 2kd = \frac{4\pi\nu d}{c}$$

$$U_{n+1} = re^{i\phi} U_n$$



$$U_1 = U_0 re^{i\phi}$$

$$U_2 = U_1 re^{i\phi} = U_0 r^2 e^{i2\phi}$$

⋮

to find total field in cavity

$$U_T = U_0 + U_1 + U_2 + U_3 \cdots$$

$$= U_0 + U_0 re^{i\phi} + U_0 r^2 e^{i2\phi} + \cdots \quad ; h = re^{i\phi}$$

$$= U_0 + hU_0 + h^2 U_0 + \cdots$$

$$= \frac{U_0}{1-h}$$

$$\boxed{U_T = \frac{U_0}{1 - re^{i\phi}}}$$ ← total field in cavity over a long averaging time

lets not lose sight of our goal: how does loss ($r<1$) effect resonance condition

took @ total intensity b/c

$$I_T = |U_T|^2$$

$$= \frac{I_0}{|1 - re^{i\phi}|^2} \quad ; \quad I_0 = |U_0|^2$$

↓ algebra

$$= \frac{I_0}{(1-r)^2 + 4r\sin^2\left(\frac{\phi}{2}\right)} \quad ; \quad \phi = 2kd = \frac{4\pi\nu d}{c}$$

ν is frequency $c = \lambda\nu$