~~Sch~~ scalar wave eqn $\left[\nabla^2 + k^2\right] u = 0$; $k = \frac{\omega}{c}$

1-) found one solution : plane wave $u = u_0 e^{ikz}$

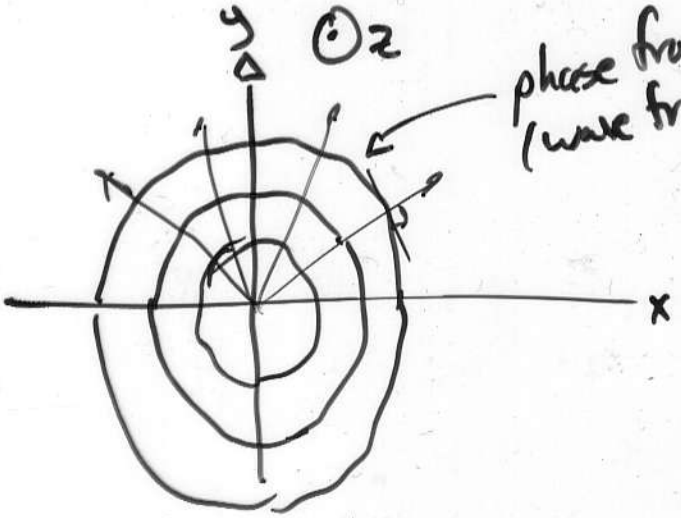
2-) another solution is the spherical wave

$u(r) = \frac{A}{r} e^{ikr}$; where r is the radial distance from the origin & A is some constant

→ The phase fronts of this wave are surfaces on constant phase

$kr = a$ ← constant

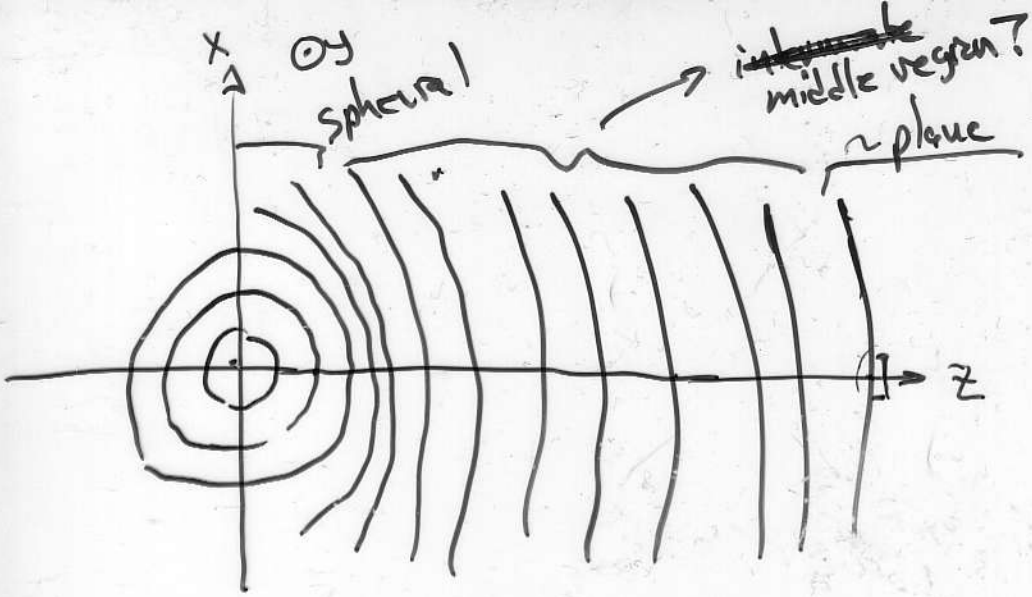
$r = \frac{\lambda}{2\pi} a$ → These are concentric spheres ; @ peaks of wave $a = 2\pi q$ $q = 0, 1, 2 \ldots$



This type of wave in optics is good for modeling light from point sources (light bulbs, etc)

for spherical wave what is the k vector & direction?

$\hat{r}$ ... why? direction of $\vec{k}$ is direction of propagation



$u = \frac{A}{r} e^{ikr}$, lets look @ propagation along z axis

→ when very far away from origin & not far off $\hat{z}$ axis. is there an approximation we can make?

→ $r = \sqrt{x^2+y^2+z^2}$

$= z\sqrt{1+\frac{x^2+y^2}{z^2}}$ → $\Delta << 1$ b/c not far off z axis

$= z\sqrt{1+\Delta}$ ← Taylor expansion

$\simeq z\left(1+\frac{1}{2}\Delta+\frac{1}{8}\Delta^2+\text{H.O.T.}\right)$

$u \simeq \frac{A}{z\left(1+\frac{x^2+y^2}{2z^2}+\frac{1}{8}\left[\frac{x^2+y^2}{z^2}\right]^2\right)} e^{ikz\left[1+\frac{x^2+y^2}{2z^2}+\frac{1}{8}\left[\frac{x^2+y^2}{z^2}\right]^2\right]}$

for amplitude term if $2z^2 \gg x^2+y^2$, then we can ignore 2nd & 3rd term of expansion

$\frac{1}{r} \simeq \frac{1}{z}$ if $2z^2 \gg x^2+y^2$ (1)

for phase term $kr \simeq kz$ if $k\left(\frac{x^2+y^2}{2z}\right) \ll \pi$ ← for it not to contribute to phase

$$2z \gg \frac{2}{\lambda}(x^2+y^2) \quad (2)$$

condition (2) much more restrictive than (1)

b/c of $\frac{1}{\lambda}$ term

in that case [of (2)] $u(r) \simeq \frac{A}{z} e^{ikz}$

... not too surprising ... plane wave b/c spherical wave to zeroth order approx (not far off z axis) looks like a plane.

.. so we have a spherical wave around origin & a plane wave @ the extreme from origin ... what in between?

lets still keep amplitude approx

$$\frac{1}{r} \simeq \frac{1}{z} \quad \text{for} \quad 2z^2 \gg x^2+y^2$$

but lets keep quadratic term in phase

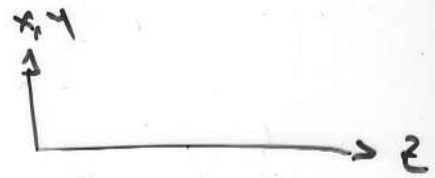
$$e^{ikr} \simeq e^{ikz\left(1+\frac{x^2+y^2}{2z^2}\right)}$$

when specifically can we make this approx?

$$\frac{z}{8}\left[\frac{x^2+y^2}{z^2}\right]^2 k \ll \pi$$ ← for 3rd term in phase expansion to add little to the phase

So now our approx of $u(r)$

$$u(r) = \frac{A}{r} e^{ikr} \simeq \frac{A}{z} e^{ikz\left(1 + \frac{x^2+y^2}{2z}\right)}$$



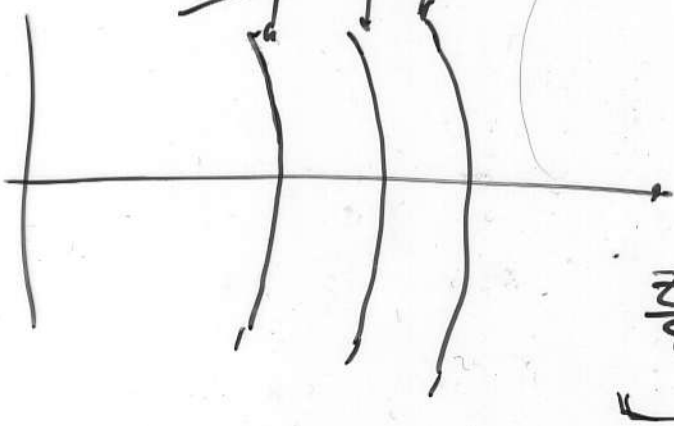
what is this physically?

$$u(r) = \underbrace{e^{ikz}}_{\text{plane wave carrier}} \underbrace{\frac{A}{z}}_{\text{amplitude correction}} \underbrace{e^{ik\left(\frac{x^2+y^2}{2z}\right)}}_{\text{radial phase}}$$

This called a ~~parabolic~~ paraboidal wave b/c the phase fronts have a parabolic function

surfaces of constant phase: $k\left(\frac{x^2+y^2}{2z}\right) = a$ ← constant ← paraboidal surface

$$z = \frac{2\pi}{\lambda} \frac{1}{2a} (x^2+y^2)$$



when can we make this approx?

$$\frac{z}{8}\left[\frac{x'+y^2}{z^2}\right]^2 k \ll \pi$$

known as the Fresnel Approx of Spherical Wave

looking @ our approx

$$u \approx \frac{A}{z} e^{ik\frac{x^2+y^2}{2z}} e^{ikz}$$

can write in a general form:

$$u(r) \approx A(r)\, e^{ikz}$$

e^{ikz} → plane wave carrier

$A(r)$ → transverse complex amplitude - contain both amp & phase info

- r means a function of (x, y, z)

we can just approx ---

$$A(r) = \frac{A_0}{z} e^{ik\frac{x^2+y^2}{2z}}$$

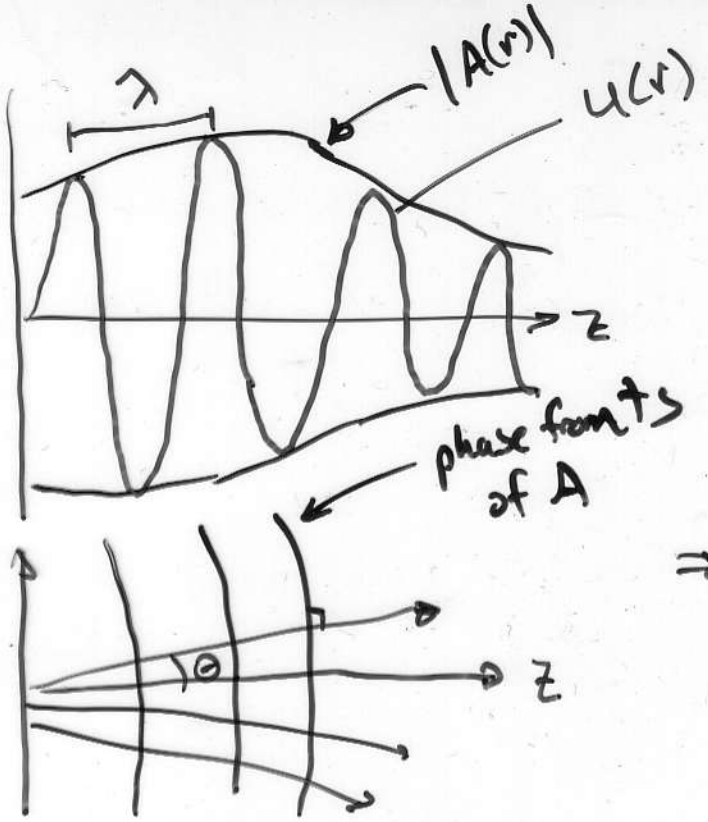
via Taylor approx on a spherical wave ... but are there other $A(r)$?

Yes, still must be E·M wave, so there must be some constraints $A(r) \Rightarrow$ needs to satisfy a wave eqn

$$\left[\nabla^2 + k^2\right] u(r)$$ ← scalar wave eqn (Helmholtz)

$$\left[\partial_x^2 + \partial_y^2 + \partial_z^2 + k^2\right] A(r) e^{ikz} = 0$$

$$\left[\partial_x^2 + \partial_y^2 + i2k\partial_z + \partial_z^2\right] A(r) = 0$$



slowly varying envelope approx

if $A(r)$ varies slowly over λ.

i.e. $\frac{\Delta A}{\Delta z} \ll \frac{A}{\lambda}$

$$\Rightarrow \frac{\partial}{\partial z} A \ll \frac{A(r)}{\lambda} = kA(r)$$

$$\frac{\partial^2}{\partial z^2} A \ll k \frac{\partial A(r)}{\partial z}$$

$$\Rightarrow \boxed{\left[\partial_x + \partial_y^2 + i 2k \frac{\partial}{\partial z}\right] A(r) = 0}$$

This called paraxial wave eqn. gives conditions on $A(r)$ so that $u(r) = A(r) e^{ikz}$ is still an E.M wave

(called paraxial) wave eqn b/c paraxial waves have k vectors that are mostly along axis

$\Rightarrow$ paraxial waves $k_z \gg k_x, k_y$