* Homework due this Friday (tomorrow) 5 pm
* I emailed a correction to HW #2 yesterday

Re-cap of paraxial wave equation

started $[\nabla^2 + k^2]u(r) = 0$ ← scalar wave eqn

$u(r) = A(r)e^{ikz}$

e^{ikz} — plane wave carrier

$A(r)$ — complex transverse amplitude

wave eqn became

$[\partial_x^2 + \partial_y^2 + i2k\partial_z + \partial_z^2]A(r) = 0$ ← no approx yet

if $A(r)$ varies slowly over λ in the z direction

$$\frac{\Delta A}{\Delta z} << \frac{A}{\lambda} \simeq kA$$

$$\frac{\partial A}{\partial z} << kA$$

$$\frac{\partial^2 A}{\partial z^2} << k\frac{\partial A}{\partial z}$$

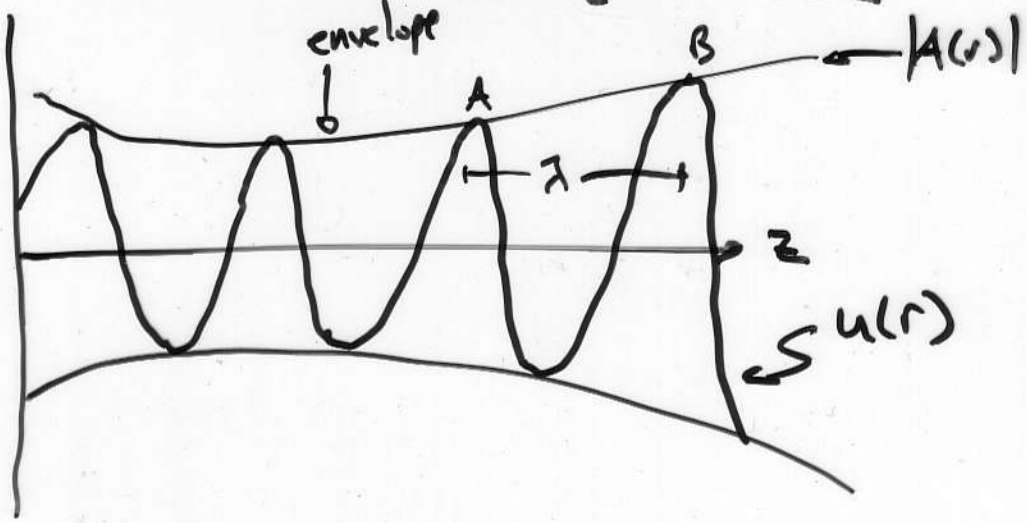
$$\Rightarrow \boxed{\left[\partial_x^2 + \partial_y^2 + i2k\frac{\partial}{\partial z}\right]A(r) = 0}$$

paraxial wave eqn

↳ gives restrictions on $A(r)$ such $u(r)$ is still an E.M

Physical picture of approx $\frac{\partial^2 A}{\partial z^2} \ll k \frac{\partial A}{\partial z}$

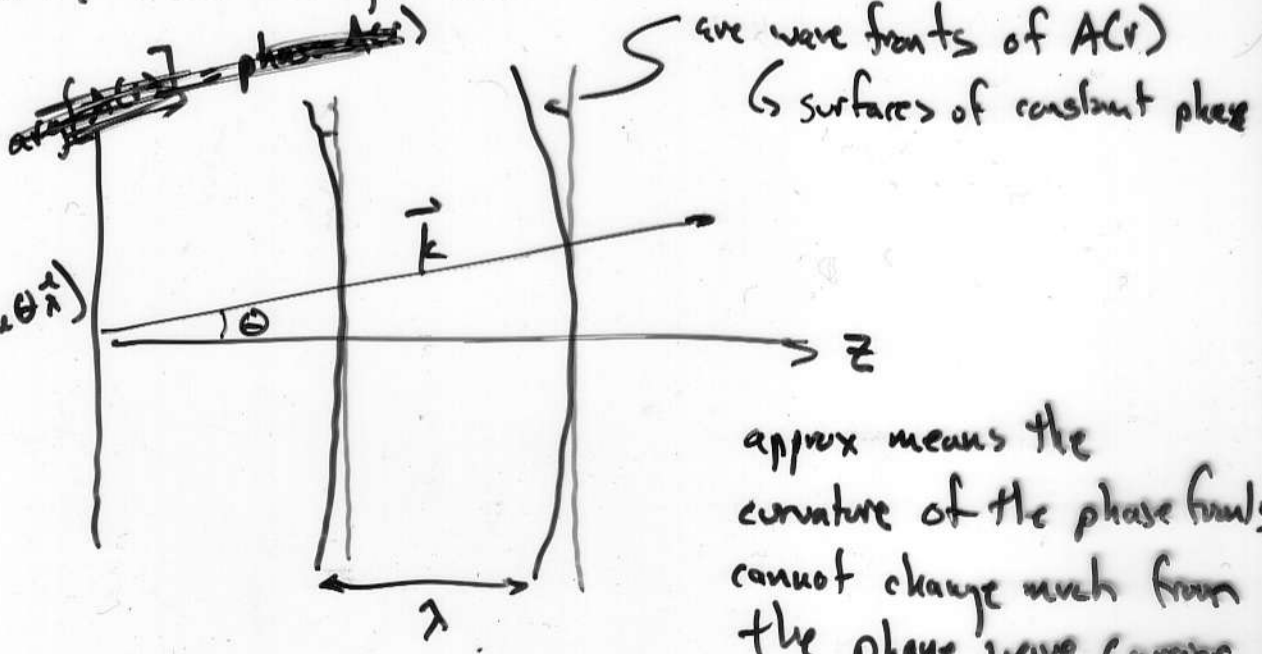
amp



approx means

$|A(r)|$ doesn't change much from A to B

phase



in 2D

$\vec{k} = k(\cos\theta \hat{z} + \sin\theta \hat{x})$

approx means the curvature of the phase fronts cannot change much from the plane wave carrier

$\Rightarrow$ b/c curvature ~~is~~ differs only slightly from the plane wave carrier, the $\vec{k}$ vector of this wave $u(r)$ are mostly in the $\hat{z}$ direction

$\rightarrow \theta \ll 1, \quad k_x, k_y \ll k_z$

paraxial waves b/c $\theta \ll 1$ small, $\Rightarrow k_z \gg k_x, k_y$

if we write the paraxial wave eqn

$$\left[\nabla_T^2 + i2k\frac{\partial}{\partial z}\right] A(r) = 0 \qquad (1)$$

2-D Schrödinger wave eqn

$$-\frac{\hbar^2}{2m}\nabla_T^2\psi + V(x,y)\psi = i\hbar\frac{\partial}{\partial t}\psi$$

$$\nabla_T^2\psi - \frac{2m}{\hbar^2}V(x,y)\psi + i\frac{2m}{\hbar}\frac{\partial}{\partial t}\psi = 0 \qquad (2)$$

$\psi \Longleftrightarrow A$

$k \Longleftrightarrow m/\hbar$

$z \Longleftrightarrow -t$

$V(x,y)$ not in (1) b/c we considered free space if it was in (1) it would correspond to a transverse refractive index

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One solution of paraxial wave eqn is

$$A(r) = \frac{A_0}{z}e^{ik\left(\frac{x^2+y^2}{2z}\right)}$$

→ The paraboidal wave that we found from an approx of the spherical wave

Thus far we've built up a suite of solutions to wave eqns that E&M used

① $u(r) = u_0 e^{ikz}$

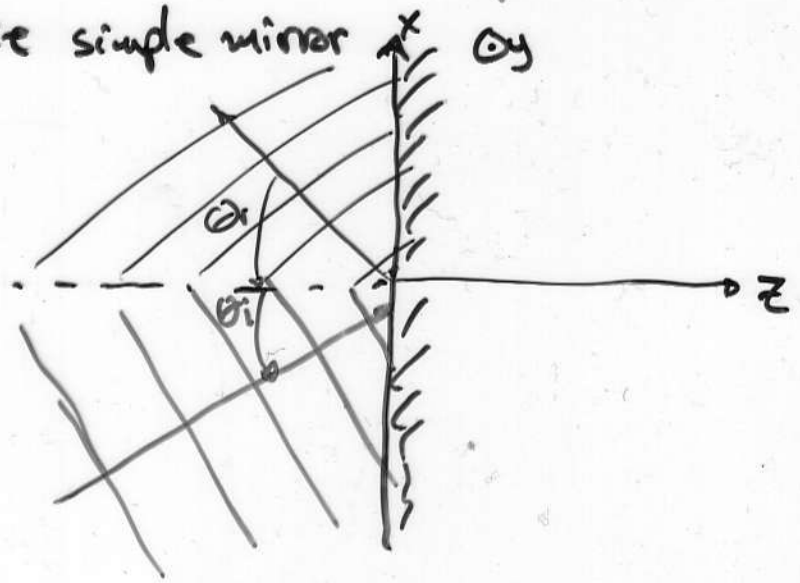
② $u(r) = \frac{u_0}{r}e^{ikr}$

③ $u(r) = \frac{A_0}{z}e^{ikz}e^{ik\frac{x^2+y^2}{2z}}$

we are going derived one more solution:
Gaussian beam later today or Tuesday

... but first some physics w/ waves through simple optical components

lets take simple mirror



plane reflecting off a mirror... what happens?

incident field

$\vec{k}_i = k_0 \cos\theta_i \hat{z} + k_0 \sin\theta_i \hat{x}$

$\vec{k}_r = k_0(-\cos\theta_R \hat{z} + \sin\theta_R \hat{x})$

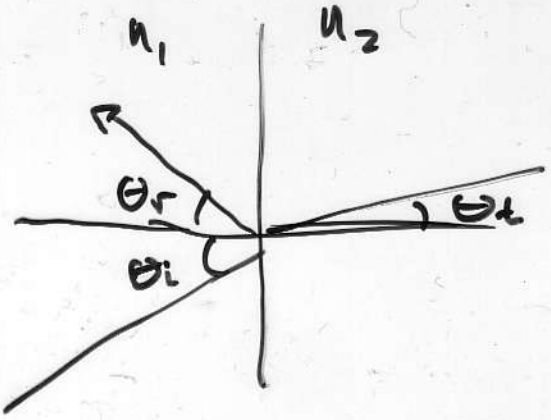
for this to be a physical situation, the phase fronts @ the mirror must continuous (no phase jumps b/w incident field & reflected field

$$\Rightarrow \vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} \quad (r\{x,y,z=0\})$$

$$k_{x_i} = k_{x_r}$$

$$\sin\theta_i = \sin\theta_r$$

$$\theta_i = \theta_r \rightarrow \text{"law of reflection"}$$



by same arguement of continuous phase fronts

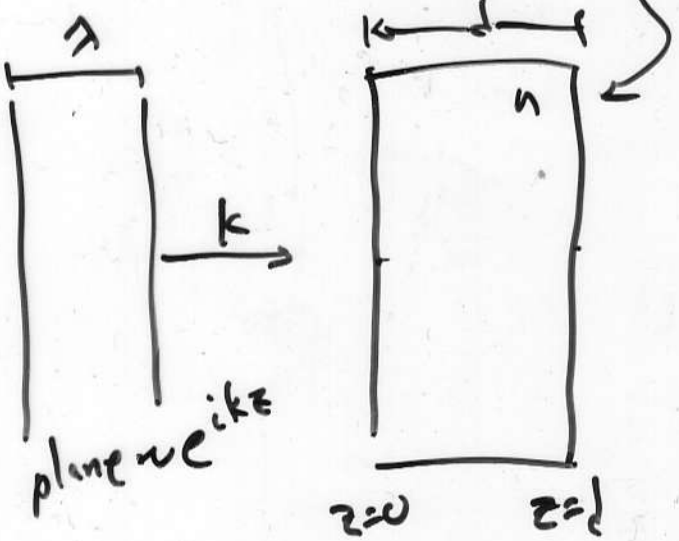
$$k_{xi} = k_{xt}$$

$$|k_0| n_1 \sin\theta_i = |k_0| n_2 \sin\theta_t$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$

Snell's law

will treat change in amplitude on r & t fields later b/c polarization matters



What we are worried about w/ scalar theory is phase that is imparted on the wave

$$u(z=d) = t_1 u(z=0)$$

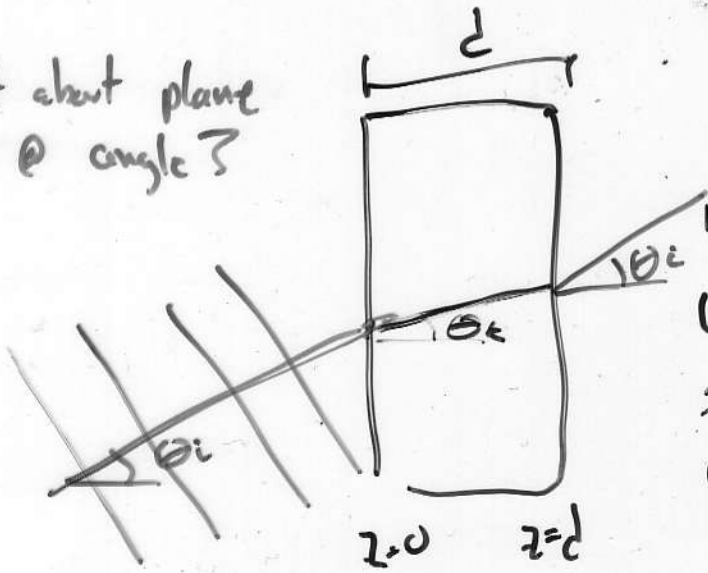
↑ transmittance of a transparent plate

↳ no loss

$$t_1 = e^{ink_0 d}$$

$$k_0 = \frac{2\pi}{\lambda_0}$$ ↳ vacuum wavelength

what about plane wave @ angle?



inside plate, field looks like

$$u = u_0 e^{ikn\frac{1}{2}(z\cos\theta_t \hat{z} + x\sin\theta_t)}$$

so after plate...

$$u(z=d) = t_2 u(z=0)$$

$$t_2 = \frac{u(x,y,z=\ell)}{u(x,y,z=0)} = e^{ikn_0 d\cos\theta}$$

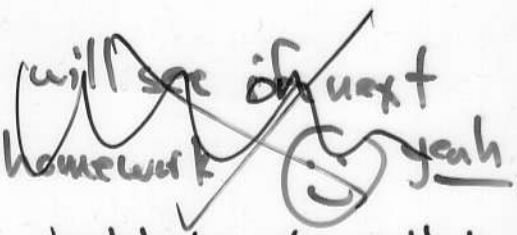
$$t_2 = \frac{u_0 e^{ikn(d\cos\theta_2 + x\sin\theta_2 + d\tan\theta_2)}}{u_0 e^{ikn(\sin\theta x)}}$$



$\tan\theta = \frac{x}{d}$

$x = d\tan\theta$

$t_2 = e^{ikn d\cos\theta}$ → why?



for paraxial waves

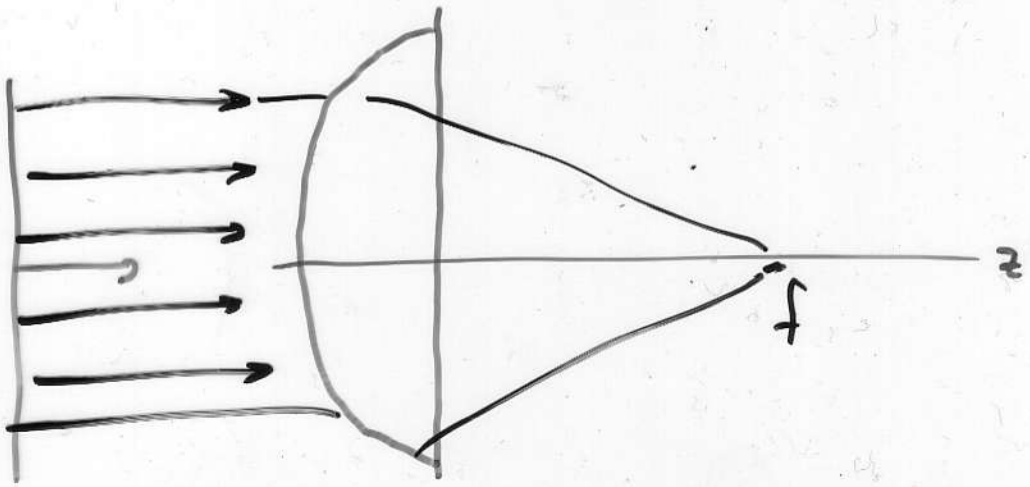
$\theta \ll 1$, $\cos\theta \approx 1$

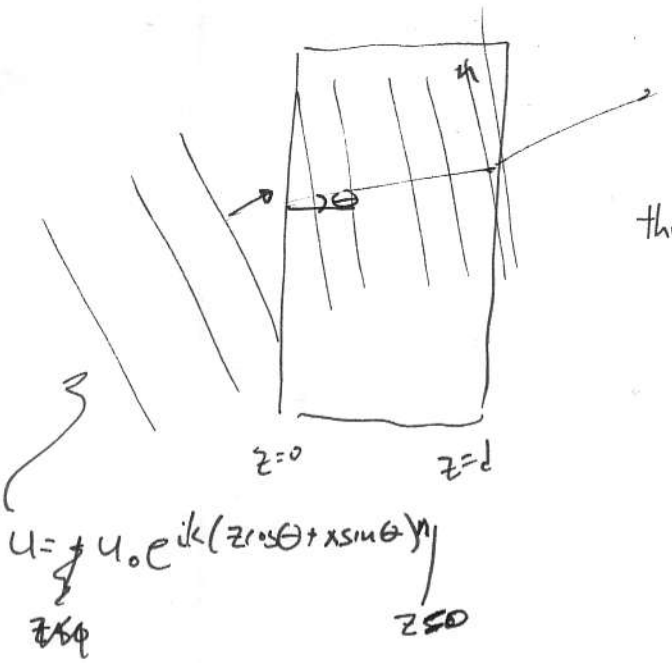
for paraxial

$t_2 = e^{ikn d}$ for transparent thin plates w/ small angles

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Okay lets do something a bit more interesting





$u = u_0 e^{ik(z\cos\theta + x\sin\theta)n}$ for $z \leq 0$

w/in glass piece field has form:

$u = u_0 e^{ik(z\cos\theta + x\sin\theta)}$ / $0 < z < d$

thus writing

$$\frac{u(x,y,z=d)}{u(x,y,z=0)} = t_2(x,y)$$ ← transmittivity of plate

$$\frac{u_0 e^{ik(d\cos\theta + x\sin\theta)n}}{u_0 e^{ik(z x\sin\theta)}} = t_2(x,y)$$ ← stays as x b/c you will have a variation in the wave as you move along x. @ $z = d$

$$\Rightarrow \boxed{t_2(x,y) = e^{ikd\cos\theta n}}$$

if $\theta \ll 1$, Then $\cos\theta \approx 1 - \frac{\theta^2}{2}$

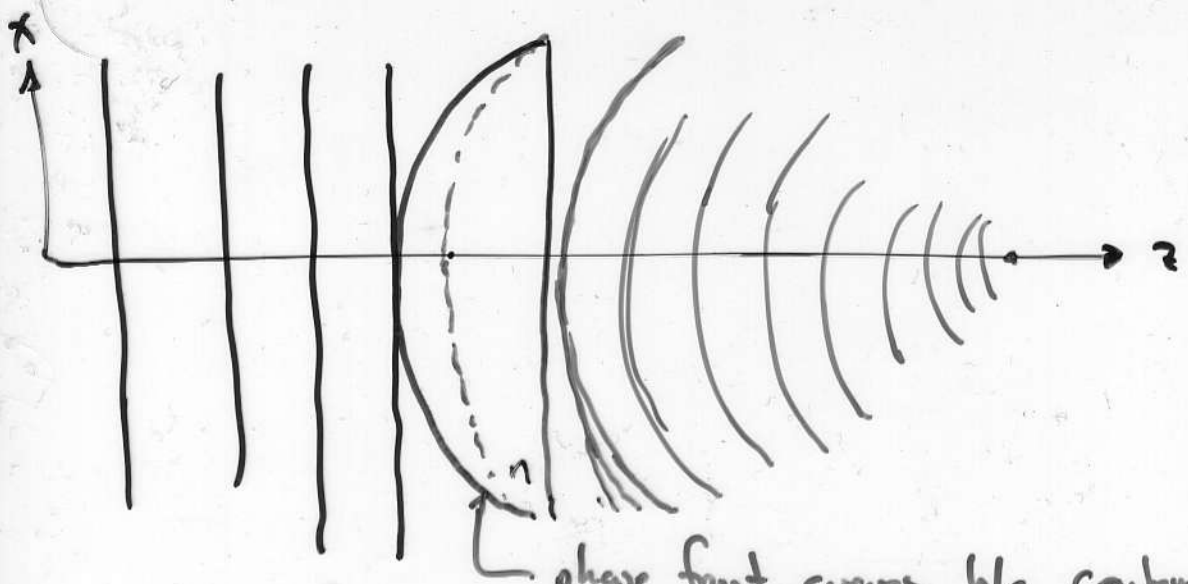
$$t_2(x,y) = e^{ink_0 d} \exp ik_0 \frac{\theta^2}{2} dn$$

if plate is Thin enough, ~~Then~~ still $\theta \ll 1$, then

$k_0 \frac{\theta^2}{2} dn \ll 2\pi$, Then The second phase term can be ignored:

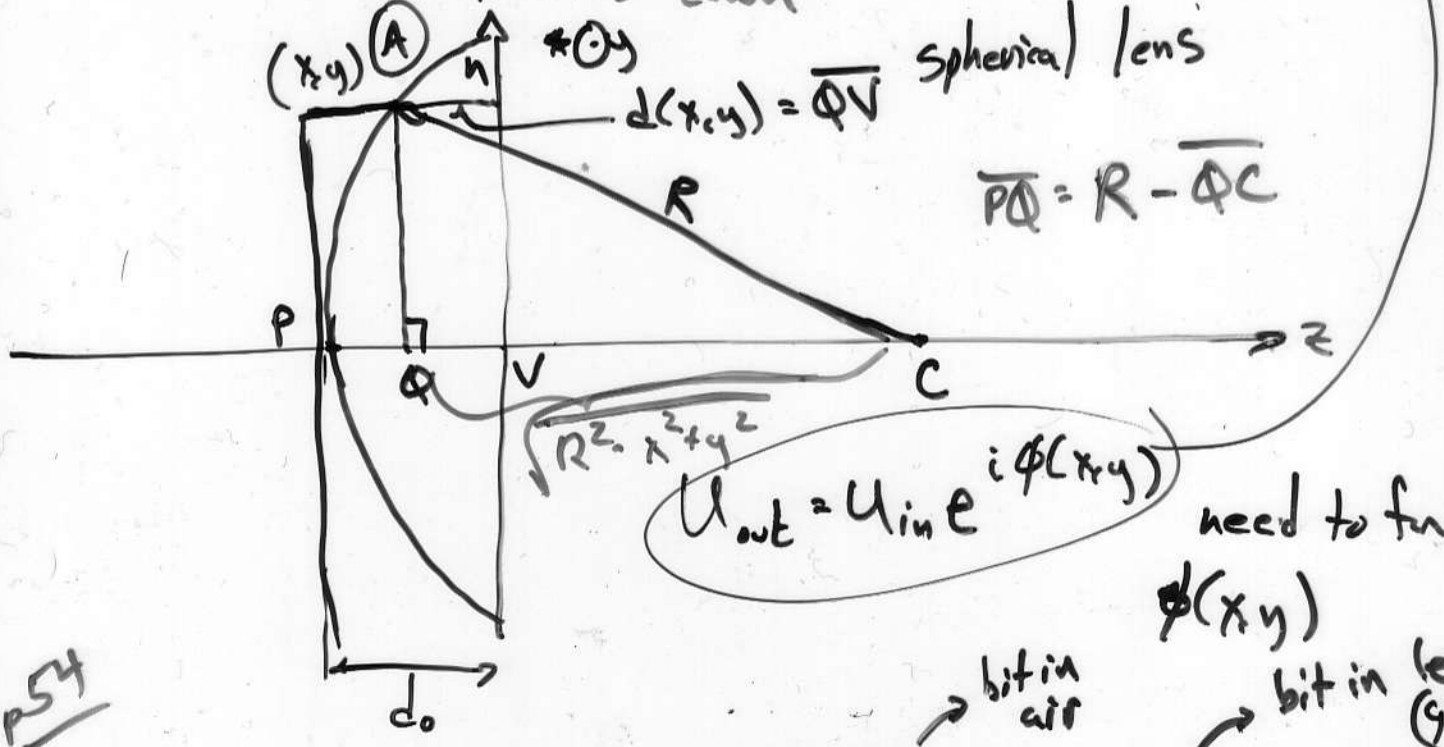
$$\boxed{t_2 \approx e^{iknd}d}$$

go to wave model:



in writing this we are assuming the lens is thin so we can ignore refractive effects

phase front curves b/c centers of wave slows down



spherical lens

$\overline{PQ} = R - \overline{QC}$

$U_{out} = U_{in} e^{i\phi(x,y)}$ need to find $\phi(x,y)$

p54

@ point A $\phi(x,y) = ik_o[\overbrace{d_o - d(x,y)}^{\text{bit in air}} + \overbrace{n\,d(x,y)}^{\text{bit in lens (glass)}}]$

$= ik_o[d_o + (n-1)\,d(x,y)]$

$d(x,y) = d_o - \overline{PQ} = [d_o - R - \sqrt{R^2 - (x^2+y^2)}]$

We don't like $\sqrt{\ }$ can we approx?

Why yes... if $x^2+y^2 < R^2$ ← small distances off z axis

$$\sqrt{R^2 - x^2+y^2} = R\left[1 - \frac{x^2+y^2}{R^2}\right]^{1/2}$$

binomial expansion

$$\simeq R\left[1 - \frac{x^2+y^2}{2R^2}\right]$$

plugging back into $d(x,y)$

$$\Rightarrow d(x,y) \simeq d_0 - \frac{x^2+y^2}{2R}$$

$$U_{out} = U_{in} e^{ik_0\left[d_0 + (n-1)\left\{d_0 - \frac{x^2+y^2}{2R}\right\}\right]}$$

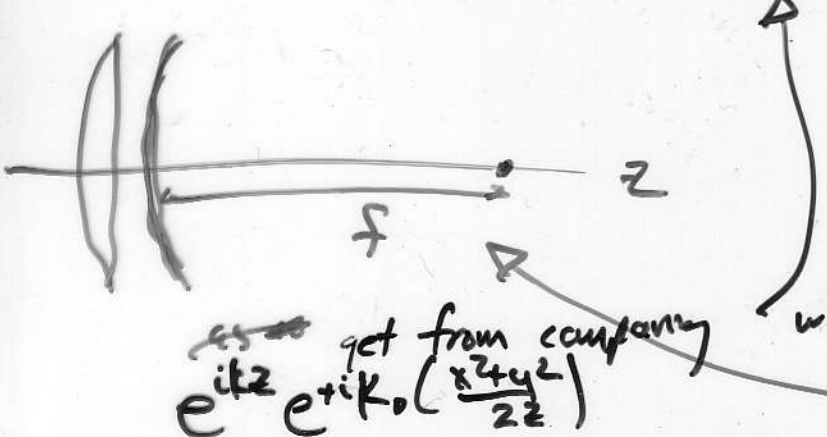
$$= U_{in} h_0 e^{-ik_0\left[\frac{x^2+y^2}{\frac{2R}{n-1}}\right]}$$

where h_0 is some constant, ~~no sqrt~~ please w/ no symmetry

lets call $\frac{R}{n-1} \equiv f$

$$U_{out} = U_{in} h_0 e^{-ik_0\left(\frac{x^2+y^2}{2f}\right)}$$

$$= U_0 e^{ikz} h_0 e^{-ik_0\left(\frac{x^2+y^2}{2f}\right)}$$



upon exiting the lens, the wave has a paraboidal form & it appears to be originating a distance f away