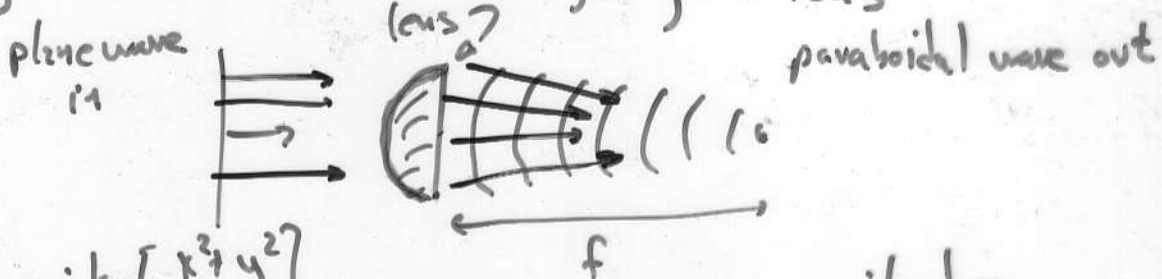
* In class Midterm Feb 28th (Thurs after spring break)
  - One hour
  - Closed book
  - I will provide formulae sheet beforehand

On thursday we ~~left~~ finished analyzing a lens

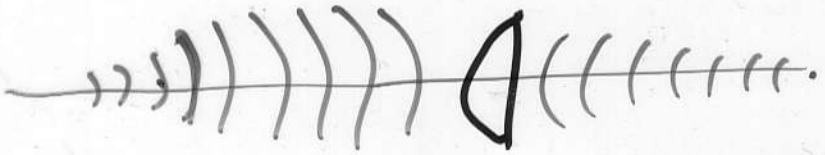


$$U_{out} = U_{in} e^{-ik_0\left[\frac{x^2+y^2}{2f}\right]} h_0 \; ; \quad f = \frac{R}{n-1} \; ; \; h_0 = e^{ik_0 d_0 n}$$

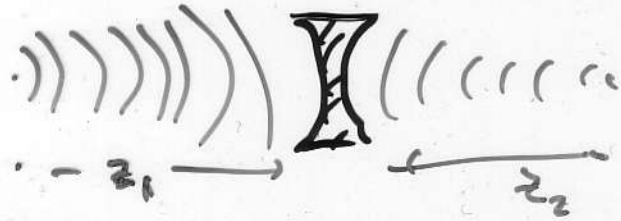
$$\Rightarrow t(x,y) = \frac{U_{out}}{U_{in}} = h_0 e^{-ik_0\left(\frac{x^2+y^2}{2f}\right)}$$ transmittance of a lens

so that even U_{in} isn't a plane wave we can still use $t(x,y)$ to see what happens to the output wave

can have any input



for your homework



- find $t(x,y)$
- see where z_2 is

What assumptions have we made in deriving

3) ~~paraxial~~ 1) thin lens → length of ≈ length of

2) monochromatic (true for every 408 class)

3) paraxial waves

→ approx spherical surface of lens w/ ~~paraxi~~ paraboloid
i.e. binomial expansion of lens curvature

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list of waves that we have covered

plane wave $E_0 e^{ikz} = u(z) = u(r)$

spherical wave $\frac{A_0}{r} e^{ikr} = u(r)$

paraboidal wave $\frac{A}{z} e^{ikz} e^{i\frac{x^2+y^2}{2z}k} = u(r)$

- but we still don't have a wave w/ finite transverse amplitude (like laser)
- How to find one?

$u(r) = A(r) e^{ikz}$ w/ $A(r)$ satisfies the paraxial wave eqn

Then $u(r)$ is an E·M wave (in paraxial approx of course)

(1) $\left[\nabla_T^2 + i2k\frac{\partial}{\partial z}\right] A = 0$ → w/ $A(r)$ obeying the slowly varying envelop approx...

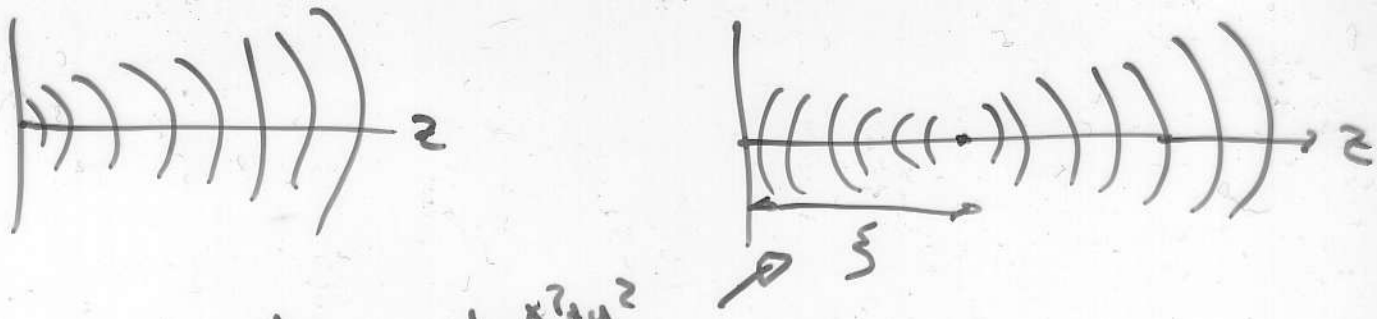
we found that one solution

$A(r) = \frac{A}{z} e^{ik\frac{x^2+y^2}{2z}}$

$z \to z - \xi$ where ξ is some constant then

$A(r) \to A(x, y, z-\xi)$ still is a solution to (1)

$A(r) \Rightarrow A(x, y, z-\xi)$



$$A(r) = \frac{A}{z-\xi} e^{ik\frac{x^2+y^2}{2(z-\xi)}}$$

... what $\xi \equiv iz_0$ i.e. ξ is purely imaginary

→ still a solution to (1) ... still E-M wave

gives us a transverse amplitude dependence

$$A(r) = \frac{A}{z-iz_0} e^{ik\frac{x^2+y^2}{2(z-iz_0)}\frac{(z+iz_0)}{(z+iz_0)}} \qquad x^2+y^2 = \rho^2$$

$$= \frac{A}{z-iz_0} e^{ik\frac{\rho^2(z+iz_0)}{2(z^2+z_0^2)}}$$

$$A(r) = \underbrace{\frac{A(z+iz_0)}{z-iz_0(z+iz_0)}}_{\text{longitudinal term}} \underbrace{e^{ik\frac{\rho^2 z}{2(z^2+z_0^2)}}}_{\text{radially dependent phase term}} \underbrace{e^{-\frac{kz_0\rho^2}{2(z^2+z_0^2)}}}_{\text{radially dependent amplitude term}}$$

longitudinal term ($\sim z$) has a phase & amplitude term

so by shifting $z \to z - iz_0$ where z_0 is some real constant we have an ~~analytic~~ analytic solution of paraxial wave eqn that has finite transverse ~~amp~~ amplitude.

what exactly is this wave ? how does it behave?

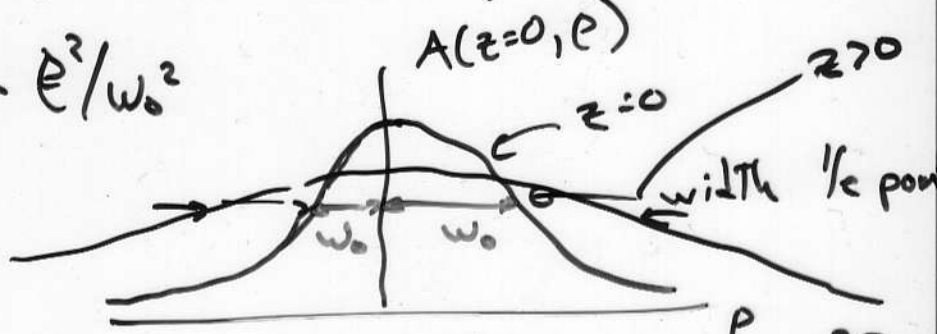
lets look @ $z=0$

$$A(r)\Big|_{z=0} = \frac{iA}{z_0} e^{-\frac{k z_0 \rho^2}{2 z_0^2}}$$

$\frac{iA}{z_0}$ → amplitude term (constant)

$e^{-\frac{k z_0 \rho^2}{2 z_0^2}}$ → transverse depend amp

$$= \frac{iA}{z_0} e^{-\rho^2/w_0^2}$$



1/e point $1 = \frac{k}{2 z_0} \rho^2$; $k = \frac{2\pi}{\lambda}$

b/c transverse dependance is gaussian ... we call this a gaussian beam.

$$\Rightarrow \rho = \pm \sqrt{\frac{\lambda z_0}{\pi}} \equiv w_0$$

Note $z=0$, no transverse phase ... so @ $z=0$ phase front is flat (look in that regard) like a plane wave.

what happens @ some $z>0$

look @ the three terms

1-) transverse amplitude term

$$A(r)\Big|_{z>0} \sim e^{-\frac{k z_0 \rho^2}{2(z^2+z_0^2)}} : e^{-\frac{\rho^2}{2\left(\frac{z^2+z_0^2}{z_0 k}\right)}}$$

- still a gaussian ... but a bigger width

to help process the math & figure out the behavior of the gaussian beam w/ distance, it is useful to define $q(z) \equiv z - iz_0$ → "q parameter"

then $\frac{1}{q(z)} = \frac{1}{z - iz_0} = \frac{z + iz_0}{z^2 + z_0^2}$

$$\equiv \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}$$

$R(z)$ is phase front curvature

$w(z)$ is transverse amplitude width

lets us rewrite the gaussian in ~~fx~~ friendly manner

$$\boxed{u(r) = A\frac{w_0}{w(z)} e^{-\rho^2/w^2(z)} e^{-ik\rho^2/2R(z)} e^{ikz} e^{-i\zeta(z)}}$$

$\rho^2 = x^2 + y^2$ → radial distance off z axis

$A = \frac{A_0}{iz_0}$ → overall amplitude constant → only two free parameters

z_0 → constant ~~known~~ known as Rayleigh range

$w_0 = \sqrt{\frac{\lambda z_0}{\pi}}$ → minimum beam radius ($z=0$)

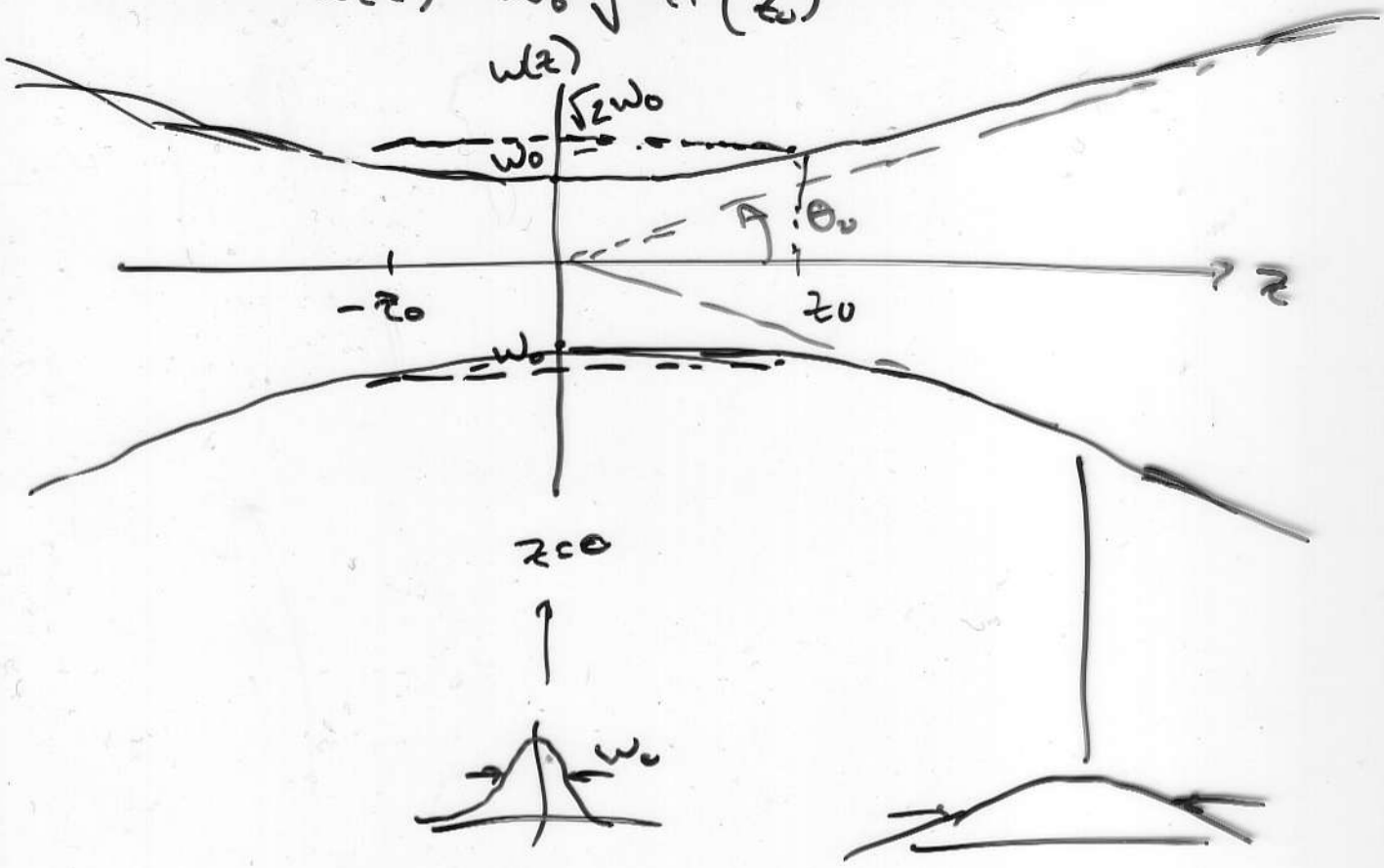
$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}$ → beam radius as a function of z

$R(z) = z\left[1 + \left[\frac{z_0}{z}\right]^2\right]$ → transverse phase front curvature

$\zeta(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$ → longitudinal phase correction from a plane wave

beam width as a function of z

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$



we can see what z_0 is physically .. it is the distance over which the beam doesn't expand much $\Rightarrow$ $2z_0$ known as the depth of focus.

for $z \gg z_0$, then $w(z) \simeq \frac{w_0}{z_0} z \equiv \theta_0 z$

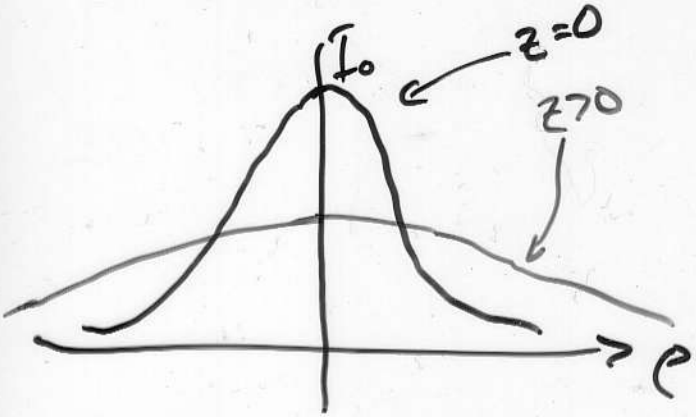
θ_0 is the divergence angle ~ linear of the ~~beam~~ beam ← need for homework

Intensity profile

$$I(r) = |A|^2 \quad \left[\frac{W}{m^2}\right]$$

$$= |A|^2 \frac{w_0^2}{w^2(z)} e^{-\frac{2\rho^2}{w^2(z)}}$$

$$= I_0 \frac{w_0^2}{w^2(z)} e^{-\frac{2\rho^2}{w^2(z)}}$$



$\underbrace{\frac{I_0 w_0^2}{w^2(z)}}$ → the peak intensity has a z dependance

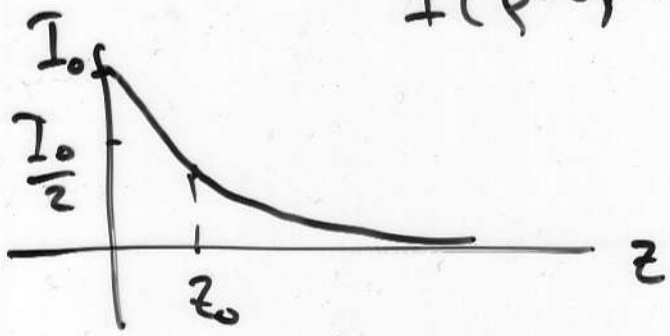
→ when $\rho=0$

so as $z>0$

→ beam wider & peak intensity decreases

- if we plot $\rho=0$ (on axis peak intensity) as function z

$$I(\rho=0, z) = I_0 \frac{w_0^2}{w^2(z)}$$



Phase

$$\phi(\rho, z) = kz - \underbrace{\zeta(z)}_{\text{Gouy phase shift}} + \underbrace{\frac{k\rho^2}{2R(z)}}_{\text{transverse phase}}$$

kz ↑ plane wave carrier

slight correction to plane wave longitudinal phase

↳ longitudinal phase

$$R(z) = z\left[1 + \left[\frac{z_0}{z}\right]^2\right]$$

points of interest

$z = 0$	$R = \infty$	plane wave
$z = \infty$	$R = \infty$	plane wave
$z = z_0$	$R = 2z_0$	- most curvature