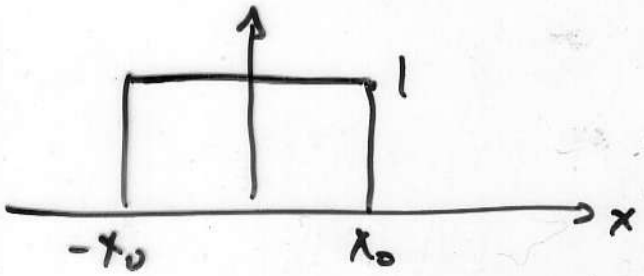
- On Thurs my lecture notes had a sign error for the definition of a spatial ~~Fourier~~ Fourier Transform
- Corrected on scanned notes now.
-

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k_x) e^{+ik_x x}\, dk_x \equiv \mathcal{F}^{-1}\left[\tilde{f}(k_x)\right]$$

$$\tilde{f}(k_x) = \int_{-\infty}^{\infty} f(x) e^{-ik_x x}\, dx \equiv \mathcal{F}\left[f(x)\right]$$

* Homework length? if you get stuck or taking too long come talk to me.

Another example of Fourier Transforms



$f(x) = 1 \quad -x_0 \le x \le x_0$

$= 0 \quad$ else

what it is $\tilde{f}(k_x)$?

$$\tilde{f}(k_x) = \int_{-\infty}^{\infty} f(x) e^{-ik_x x}\, dx = \int_{-x_0}^{x_0} e^{-ik_x x}\, dx = \frac{1}{-ik_x} e^{-ik_x x}\Big|_{-x_0}^{x_0}$$

$$= +i \frac{1}{k_x}\left[e^{-ik_x x_0} - e^{+ik_x x_0} \right]$$

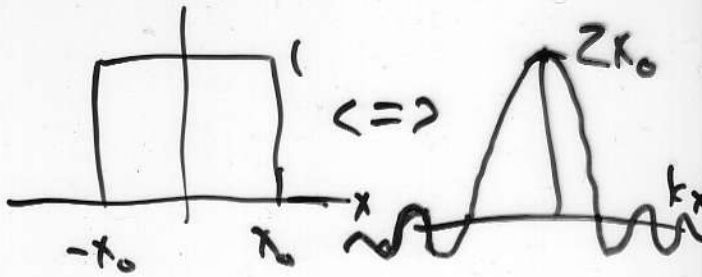
$$= i \frac{1}{k_x}\left[-i 2 \sin k_x x_0 \right]$$

$$= \frac{2}{k_x} \sin k_x x_0$$

$$= \frac{2x_0}{} \frac{\sin k_x x_0}{k_x x_0}$$

$$\frac{\sin x}{x} \equiv \operatorname{sinc} x$$

$$= 2x \operatorname{sinc}\left[k_x x_0 \right]$$



$f(x)$	$\tilde{f}(k_x)$
$\operatorname{rect}\left[\frac{x}{2x_0}\right]$	$2x_0 \operatorname{sinc}\left[k_x x_0\right]$
$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}} e^{-\left[\frac{k_x}{4\alpha}\right]^2}$ ← $e^{-\left[\frac{k_x}{4\alpha}\right]^2}$
$\delta(x)$	1
1	$2\pi\delta(k_x)$
$e^{ik_0 x}$	$2\pi\delta(k_x - k_0)$
$f(x)e^{ik_0 x}$	$\tilde{f}(k-k_0)$
$f(x-x_0)$	$\tilde{f}(k_x)e^{-ik_x x_0}$
$\cos k_0 x$	$\frac{1}{2}\left[\delta(k_x - k_0) + \delta(k+k_0)\right]$

Another mathematical concept we will use is convolution

$$f(x) \otimes g(x) \equiv \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'$$

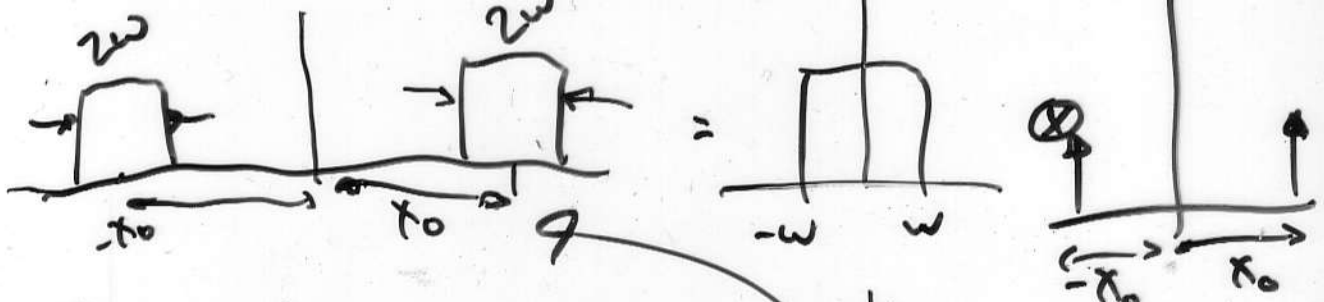
Convolution Theorem

$$\mathcal{F}\left[f(x) \otimes g(x)\right] = \mathcal{F}\left[f(x)\right] \cdot \mathcal{F}\left[g(x)\right]$$

↳ Show in online text Stack §11.3

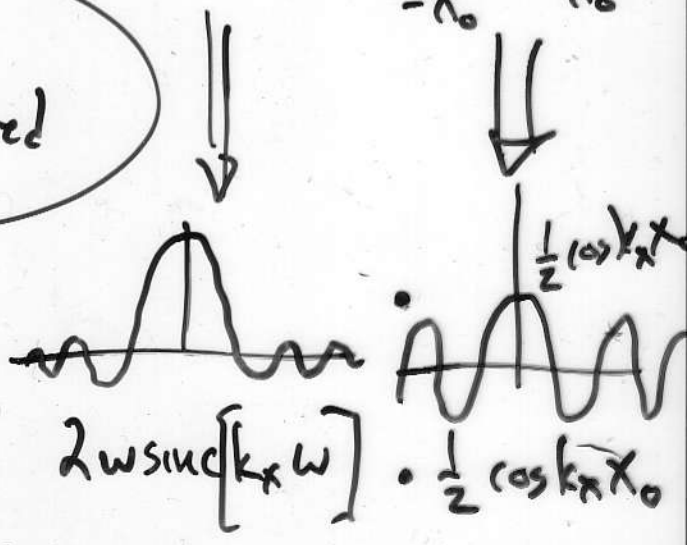
we use this we need to take $\mathcal{F}$ of a function that is complicated

$A(x) =$



what is the convol
what two functions when convolved together give me

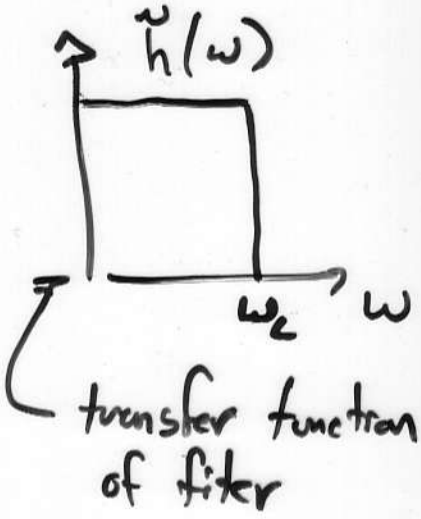
$\mathcal{F}\left(A(x)\right) =$



let's take a familiar example.. filtering of frequencies in time



→ filter low pass filter rejects any ~~frequencies~~ frequencies higher than ω_c



what is the output? $g(t)$

- Take Fourier ~~transfer~~ transform

$$\tilde{f}(\omega) = F[f(t)]$$

$$\rightarrow \tilde{f}(\omega) \cdot \tilde{h}(\omega) = \tilde{g}(\omega)$$

$\tilde{g}(\omega)$ ← output in frequency space

→ Take inverse Fourier Transform to get output in time domain

$$\Rightarrow g(x) = F^{-1}[\tilde{g}(\omega)]$$

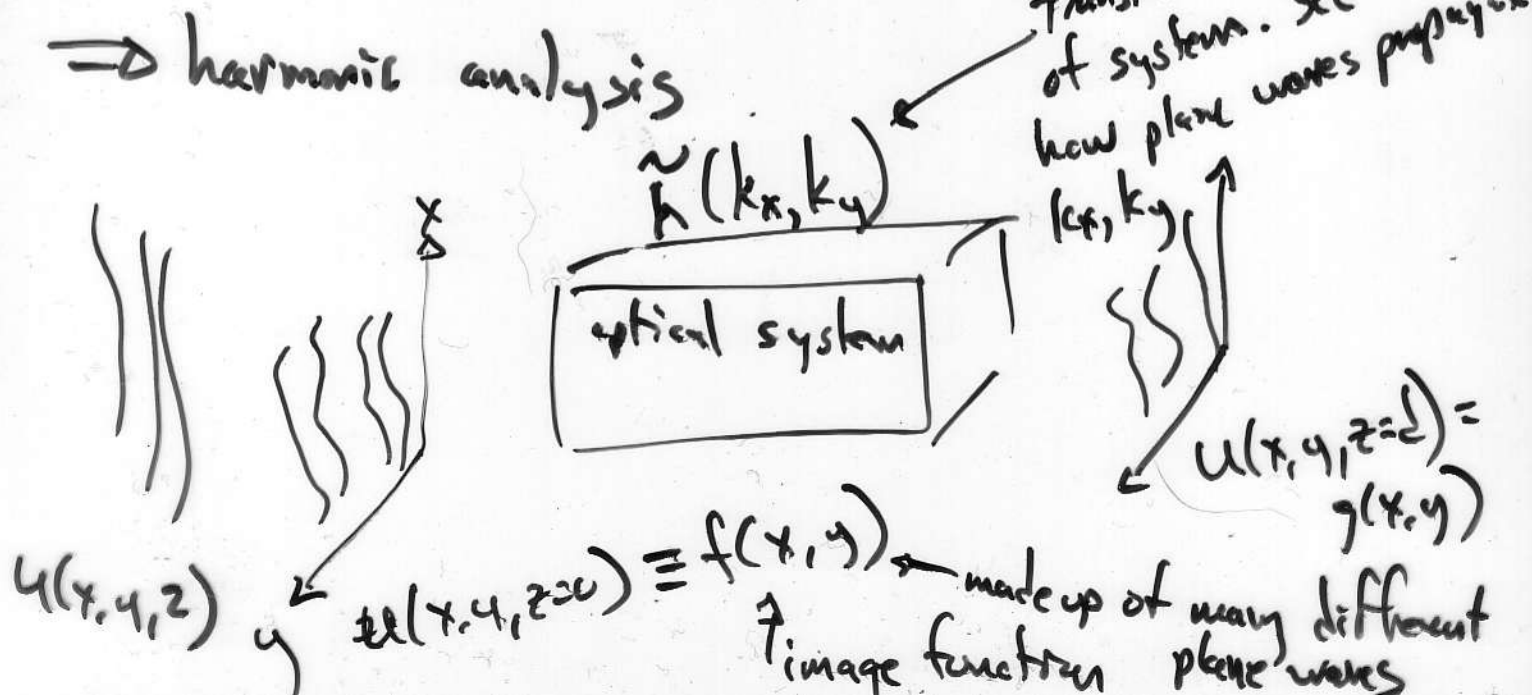
Fourier Optics... same as time example, except spatial frequencies & instead of filter we look @ how our frequencies (i.e. plane waves) respond to lens, freespace propagation, square apertures etc.

① take a general wave $u(x,y,z=0)$ split into an infite sum of plane waves

② via the transfer function we propagate these waves through our optical system

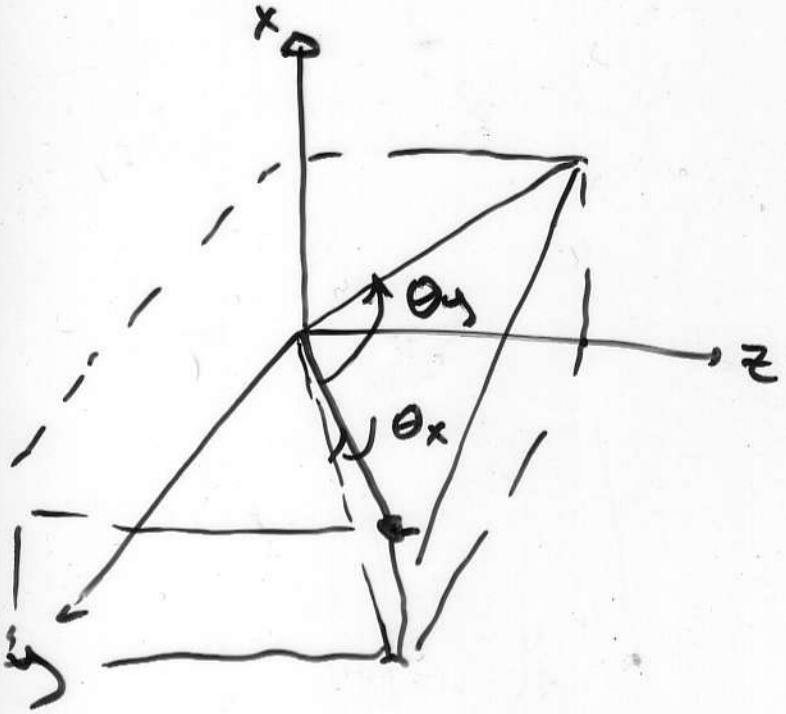
③ Sum up all resulting plane waves to get $u(x,y,z=d)$

⇒ harmonic analysis



lets get specific: simplest case

$$u(x,y,z) = e^{i(k_x x + k_y y + k_z z)} \quad \text{w/ } k_0^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$



the k vector of this wave

$$\sin\theta_x = \frac{k_x}{k} \simeq \theta_x$$

$$\sin\theta_y = \frac{k_y}{k} \simeq \theta_y$$

paraxial limit

now @ $z=0$ the image function

$$f(x,y) = u(x,y,z=0) = e^{i(k_x x + k_y y)}$$

↳ This function has spatial frequencies k_x k_y that have 1 to 1 correspondence w/ a plane wave traveling @ ~~spec~~ specific angle θ_x & θ_y in $z=0$ ~~plane~~ plane

Key to ~~Fourier~~ Fourier ~~Transform~~ Optics

↳ look @ figure 4.1-1 on page 105

moreover there is also a 1 to 1 correspondence

b/w $f(x,y)$ & $u(x,y,z)$

$$\Rightarrow u(x,y,z) = f(x,y)\, e^{ik_z z}$$

$$= f(x,y)\, e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, z}$$

Steps for our analysis

math ~~physics~~	physic ~~math~~
① Evaluate $u(x,y,z=0) = f(x,y)$	input image get
② $\tilde{f}(k_x,k_y) = \mathcal{F}[f(x,y)]$	decompose $f(x,y)$ into a sum of many plane waves each plane wave propagates @ particular angle $\theta_x \Leftrightarrow k_x$, $\theta_y \Leftrightarrow k_y$
③ $\tilde{g}(k_x,k_y) = \tilde{f}(k_x,k_y)\tilde{u}(k_x,k_y)$ (transfer function)	see effect of optical system on these plane waves
④ $g(x,y) = \mathcal{F}^{-1}[\tilde{g}(k_x,k_y)]$	sum up all plane wave @ the end to see output

so lets do free space propagation as our optical system... need to figure out transfer function is

$$\tilde{h}(k_x, k_y)$$

lets look @ our simple case w/ $f(x,y) = e^{i(k_x x + k_y y)}$

$$g(x,y) = f(x,y) e^{ik_z d}$$

$$\frac{g(x,y)}{f(x,y)} = e^{ik_z d} \equiv \tilde{h}(k_x, k_y) = e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, d}$$

↳ not always true... rather it is only true when $f(x,y)$ is a single plane wave

$$\boxed{\tilde{h}(k_x, k_y) = e^{i\sqrt{k^2 - k_x^2 - k_y^2}\, d}} \leftarrow \text{exact transfer function for free space}$$

$$\doteq e^{ikd\left(1 - \frac{k_x^2 + k_y^2}{k^2}\right)^{1/2}}$$

$$\boxed{\doteq e^{ikd\left(1 - \frac{k_x^2 + k_y^2}{2k^2}\right)}} \leftarrow \text{paraxial approx}$$

↳ Fresnel approx for freespace

$$\left[\frac{\theta_x^2 + \theta_y^2}{4\lambda}\right] d \ll 1$$

page 113 of text

So getting back to the example

@ $z=0$ $\quad f(x,y) = u(x,y,z=0)$

- decompose into all of it plane wave components

$$\tilde{f}(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,z=0)\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

- $\tilde{g}(k_x,k_y) = \tilde{h}(k_x,k_y)\,\tilde{f}(k_x,k_y)$

$=$

- Sum up to get final image

$$u(x,y,z=d) = \mathcal{F}^{-1}\left[\tilde{g}(k_x,k_y)\right]$$

Homework #5 will be due Monday after Spring Break Feb 25 2008

problem #1 change of variables to make $f(x)$ a periodic function Fourier Series