Basic Skills in Mathematics

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Abstract

What are the basic skills in Mathematics that can affect success in first-year Calculus and beyond?

We gathered data from the Basic Skills Test (BST) in Mathematics and other diagnostic tests. After analysing the predictive power of the BST with respect to performance in Differential Calculus, we isolated the questions that correlated best with course grades and grouped them based on the underlying math concepts or tasks. Our goal was to identify what basic skills are most important for success in Differential Calculus.

Furthermore, an analysis of common students' errors gave us some insight in students' misconceptions.

Basic Skills Test

- 30 questions, 1.5 hours
- Test design: 50% of questions on Algebra

     25% of questions on Functions

     25% of questions on Geometry

- Different use (depending on course)
  - placement test (full test, outside class time)
  - diagnostic test (subsets of questions, in-class/online)

Does the BST score correlate with performance in Differential Calculus?

Math 180 (2008, N= 416)

$R^2 = 0.34$
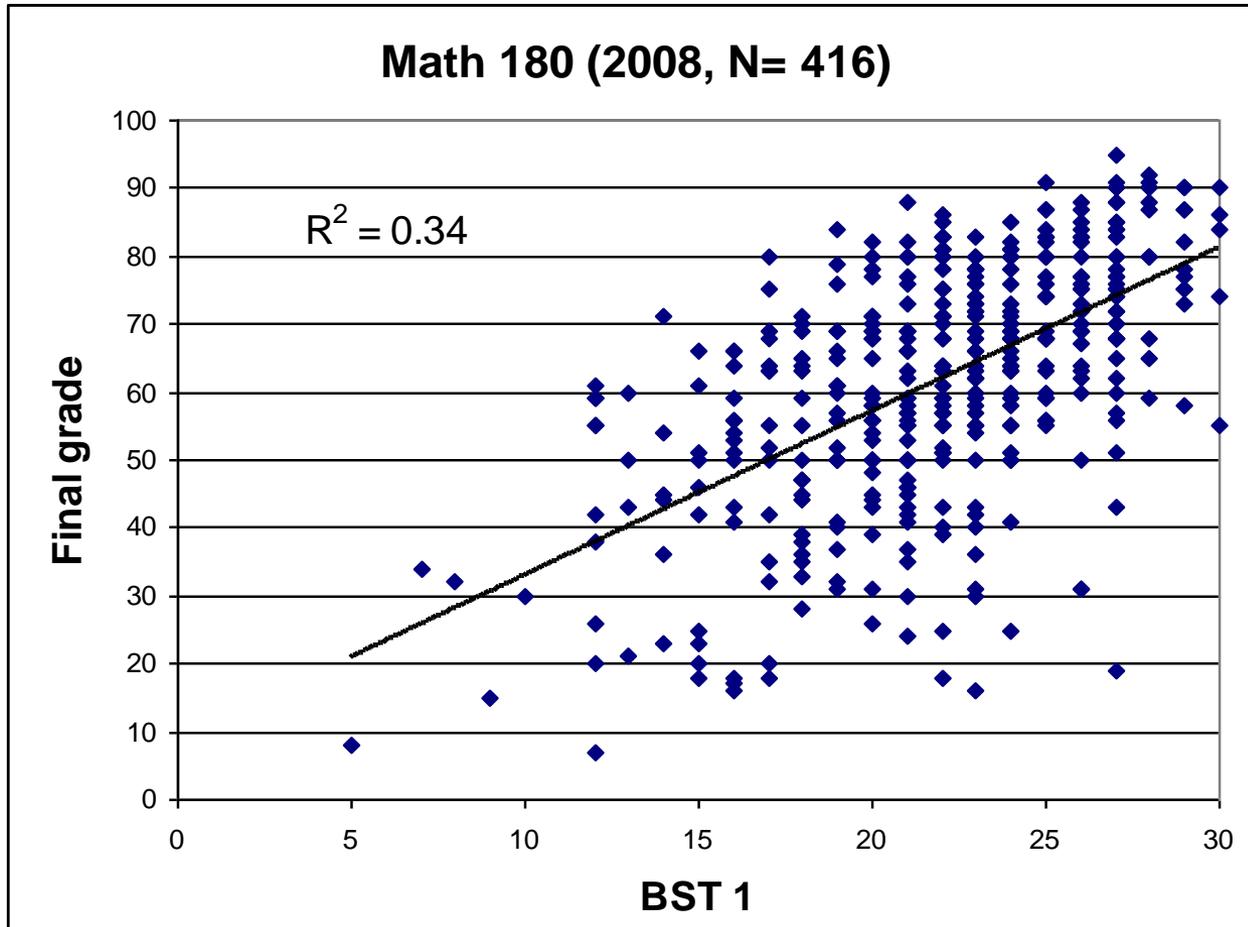
Basic skills in math are also important in other sciences

The case of Science One

(N=61)	r^2
MATH	0.72
PHYS	0.69
CHEM	0.62

PHYS

Sci1 Physics Final Mark vs BST score out of 30

MATH

Sci1 Math Final Mark vs BST score out of 30

CHEM

Sci1 Chemistry Final Mark vs BST score out of 30
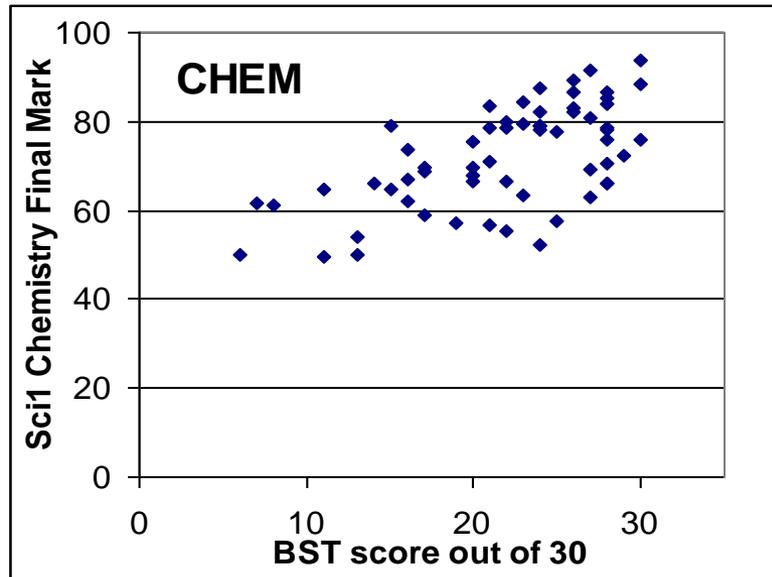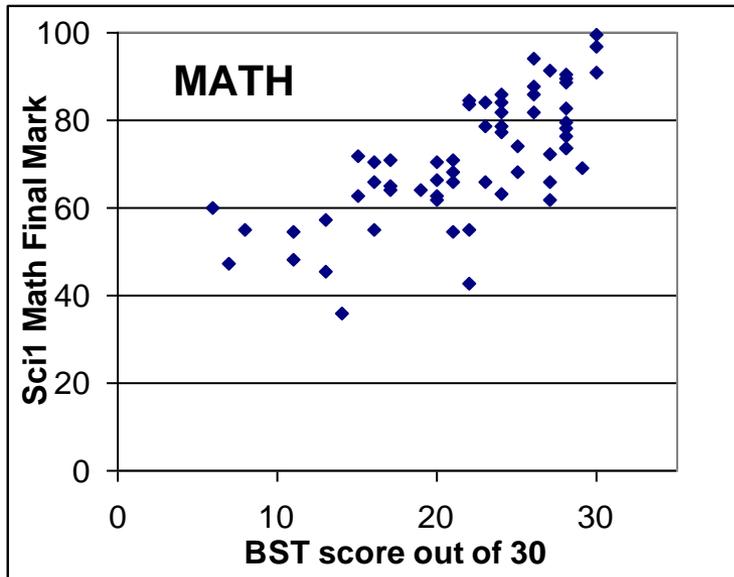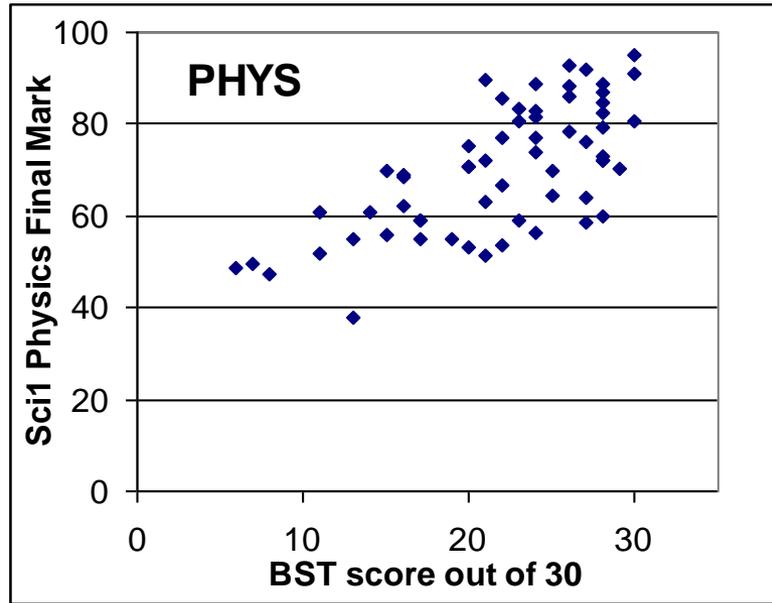
Questions that are best predictors of success in calculus

Q	9	11	12	13	14	15	17	18	20	21	23	25	27
r	0.29	0.34	0.34	0.25	0.30	0.40	0.28	0.28	0.37	0.25	0.30	0.36	0.31

r is the point bi-serial correlation coefficient

What do these questions have in common?

Main topics identified using factor analysis:

- Algebraic manipulations

- Concepts of functions and function compositions

- Word problems

Students' difficulties

Courses:

Math 220   Introduction to Mathematical Proofs

Math 104   Differential Calculus
            (for students with calculus background)

Math 184   Differential Calculus
            (for students without calculus background)

Math 110   Two-Term Differential Calculus

Algebraic manipulations

(a) isolating one variable in complex expressions

Given the expression $p = \dfrac{2q + r}{qr}$, solve for r in terms of p and q only.

Common wrong answers	Math 104	Math 184	Math 110
r is cancelled out so it is impossible to solve for *r*	8%	15%	22%
$r = 2q - pr$ or $r = \dfrac{2q+r}{qp}$	10%	15%	23%
Correct, $r = \dfrac{2q}{qp-1}$	69%	53%	29%

(b) working with inequalities

Find the domain of the function:

$$f(x) = \sqrt{x^2 - 9} \qquad\qquad f(x) = \sqrt{9 - x^2}$$

Common wrong answers	
$x \leq \pm 3,\ x \geq \pm 3$	18%
$-3 \leq x \leq 3$	20%
Correct, $x \leq -3$ or $x \geq 3$	34%

Common wrong answers	
$x \leq \pm 3,\ x \geq \pm 3$	18%
$x \leq -3$ or $x \geq 3$	8%
Correct, $-3 \leq x \leq 3$	50%

Misconceptions:

- inequalities can be solved using the same rules as for equalities

$$x^2 > 9 \quad \text{implies} \quad x > \pm 3$$

- solution of quadratic inequalities lies in the middle interval

Functions

(a) reading graphs

Consider the graph of the function $y = f(t)$ shown on the right.

If $f(-x) = 3$, find x.

x = -4	49% correct
x = 4	10%
x = -3	14%

If $f(-x) = -2$, find x.

x = 3	32% correct
x = -3	9%
x = 2	39%
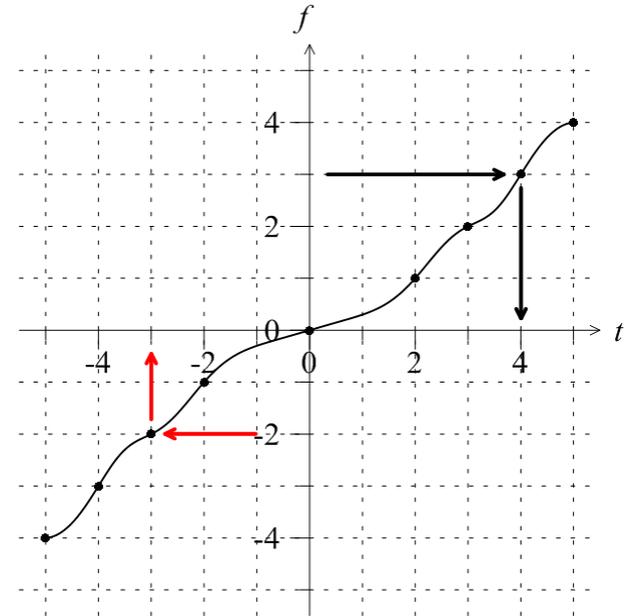
Is reading a graph in the third quadrant more challenging?

(b) function composition

- Previous exposure to calculus gives significant advantage

If $f(x) = x^4 - 3$ and $g(x) = \sqrt{x + 3}$, then the composition of functions $(f \circ g)(x)$ equals

	Correct
Math 104 (high school calculus)	55%
Math 184 (no calculus experience)	36%

Word problems

- challenging even for the top students!

Train A and B are travelling on adjacent parallel tracks. At 11 am, train A is 50 km behind train B when train A passes a grain elevator. Train A travels at a constant speed of 40 km/h and train B travels at a constant speed of 50 km/h in the same direction. Find a formula for the distance, in km, of train B from the grain elevator t hours after noon.

Common errors	Science One
Wrong initial distance or time, $d = 50 + 50t$	13%
Distance from train A, $d = 50 + (50 - 40)t$	8%
Blank / no significant progress	9%
Correct, $d = 100 + 50t$	65%

The graph of the quadratic function $y = ax^2 + bx + c$ has x-intercepts -2 and 3, and y-intercept 12. Find a, b and c.

Correct: 37% (Math 180/184), 74% (Sci1)

Solution methods

conceptual approach

Write down a system of equations… Leads to correct answers except for algebra errors

procedural approach

Write down $(x+2)(x-3) = 0$ and derive a, b, c incorrectly

Difficulties continue into second year

Find all values of x for which $|x^2 - 1| < 3$ is true.

(from diagnostic pre-test in Math 220)

Common wrong answers	Math 220
$x < \pm 2$	9%
Correctly considered two cases	33%

When proving from first principles that a given sequence converges to 1 on the final exam of Math 220 , 36% of students made erroneous inferences about inequalities.

Conclusions

Basic skills in Mathematics are an important factor for success in both math and other sciences. Three main skills were identified as best predictors of performance in calculus: fluency with algebraic manipulations, understanding of the concept of function, and working with word problems.

Students' deficiencies and common misconceptions in these skills were analysed to hopefully provide instructors with some insight on what type of help students may benefit from.

Acknowledgements