

# *Identifying and Addressing Student Difficulties with Proportional Reasoning*

Department of Physics  
University of British Columbia  
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# Acknowledgments

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*Stephen Kanim, New Mexico State University*

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*Lucas Jones and King Wong*

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# Ongoing research and curriculum development effort

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- Investigate productive and problematic aspects of students' proportional reasoning in introductory physics contexts
- Develop and assess instructional strategies to increase student facility

# Mathematical sense-making

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*Mathematization:*

The flexible and generative use of mathematics  
to describe and analyze phenomena

## Proportional Reasoning: *A foundation for concepts in physical science*

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*“The idea of ratio is at the heart of measurement. To conceive of an object as measured is to conceive of some attribute of it as segmented, and that segmentation is in comparison to some standard amount of that attribute.”*

Thompson & Saldanha, 2004

*“One of the most severe gaps in the cognitive development of students is the failure to have mastered reasoning involving ratios . . . this disability is one of the most serious impediments to their study of science.”*

– Arnold Arons,

*A Guide to Introductory  
Physics Teaching*

## Ratio and product quantities in introductory physics

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<b>Context</b>	<b>Mathematical Model</b>
Homogeneous substance	$d = M/V$
Kinematics	$v = \Delta x / \Delta t$ ; $a = \Delta v / \Delta t$
Force laws	$k = F_{\text{spring}} / \Delta x$
Periodic motion	$f = (\# \text{ of reps}) / \Delta t$
Energy	$W = F\Delta x$

*Students who have difficulty working with ratios may struggle to make sense of physics.*

# Effective coping strategy: *Abandon sense-making!*

Hammer, 1989: *Two approaches to learning physics*

Liza: Learn formulas and facts. Understanding is knowing the formulas.

Ellen: Make sense of the formalism and integrate with own intuitions.

*While Liza thrived, Ellen eventually abandoned her approach to “do it like everyone else”*

but what does sense-making look like?

*“Students . . . frequently run into formidable obstacles in the interpretation of division. They have never interpreted division as calculating how much of the numerator is associated with one chunk of whatever is in the denominator.”*

– Arnold Arons,  
*A Guide to Introductory  
Physics Teaching*

## Verbal interpretation of ratio:

- *speed* is the distance traveled during each second of the (uniform) motion
- *period* is the time required for each repetition
- *density* is the mass of each cubic cm of the homogeneous material

## *Example Problem*

The Earth's circumference is 25,000 miles. A rope is wrapped snugly around the Earth. Then 20 feet of additional rope is spliced in.

When the new, longer rope is held evenly above the Earth's surface, which animals can pass beneath it?

- a. *Amoeba*
- b. *Bumblebee*
- c. *Cat*
- d. *Dromedary*

*Interpret Pi as a ratio (and apply ratio to calculate)*

Math courses generally focus on the context-free use of number.

In physics, we must build and interpret ratios *in context*.

*Mathematization*

# Mathematical sense-making

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*Mathematization:*

The flexible and generative use of mathematics  
to describe and analyze phenomena

*Mathematization:*

A hallmark of physics for which students may  
have little preparation

# Question for Research

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*To what extent are students able to interpret ratios in context?*

# Research Methods

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*Develop and administer written questions and analyze student responses.*

*Individual student interviews are used to validate questions and probe reasoning in more depth.*

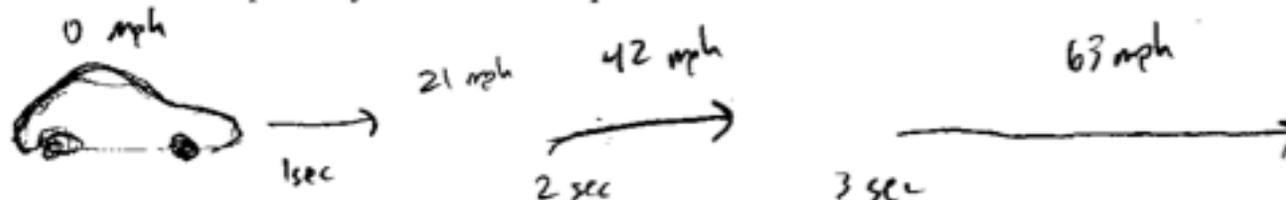
# The Porsche question

On its website, Porsche has the statement “21 mph per second” in the technical specs for one of its cars.

a. Describe the information that the number 21 gives about the car. Be as specific as you can.

*It describes that the car is increasing its velocity by 21 mph for each second that goes by... the acceleration.*

b. Make a sketch that explains your answer to part a.



Administered to:

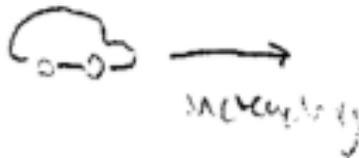
- WWU calc-based mechanics, ungraded course pretest, N = 59
  - ❖ ~ **65% provide a complete interpretation**
- NMSU calc-based mechanics, course exam, N = 76
  - ❖ ~ **45% provide a complete interpretation**

## Failure to provide an interpretation:

- a. Describe the information that the number 21 gives about the car. Be as specific as you can.

That is rate of acceleration

- b. Make a sketch that explains your answer to part a.



## Failure to provide an interpretation:

*“Miles per hour per seconds. Acceleration.”*

*“21 is the acceleration because you are given a distance/time<sup>2</sup>”*

*“Seems like an acceleration (distance/time/time), but hr are not equal to sec”*

*“21 mph per second describes the acceleration. ‘21 mph’ is a velocity; when ‘per second’ is added to that velocity statement, it turns into an acceleration so that the units are 21 mi/hr\*s”*

## **Incomplete or muddled interpretation:**

*“The car has an acceleration of 21 mph every 1 second.”*

*“21 is how fast the car accelerates in 1 second.”*

*“21 is the acceleration the car can attain in 1 second.”*

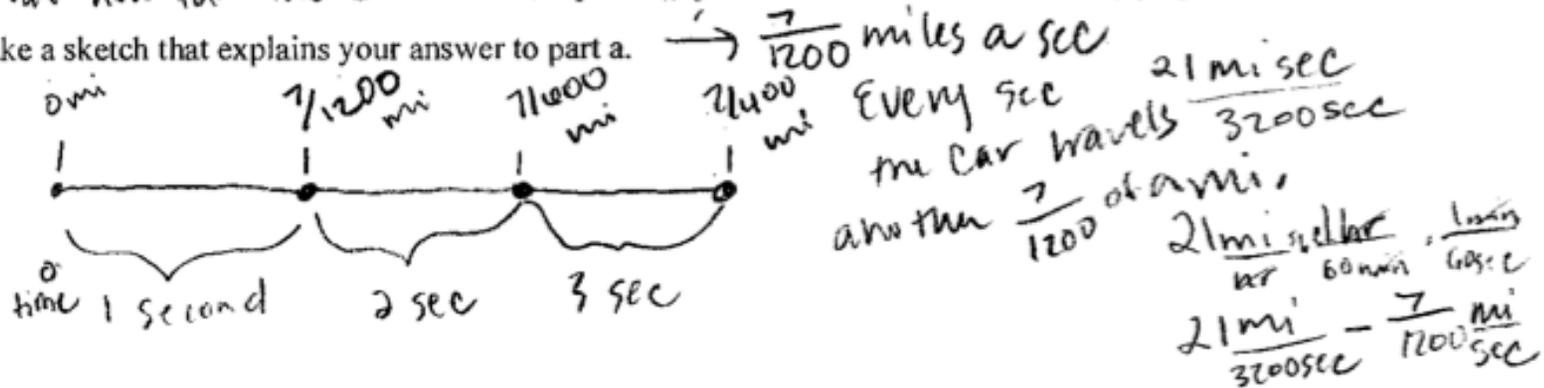
# Incorrect (confusion btwn vel and accel):

On its website, Porsche has the statement "21 mph per second" in the technical specs for one of its cars.  $21 \frac{\text{mi}}{\text{hr}} \frac{1}{\text{sec}}$

- a. Describe the information that the number 21 gives about the car. Be as specific as you can.

21 miles per hour per second. The number 21 is giving a distance about how far the car travels but the units are not the same.

- b. Make a sketch that explains your answer to part a.



## Incorrect (difficulties with "per"):

By this, Porsche means that the particular model is characterized as the number of miles per hour the car can go in one second. In this case the number of miles per hour is 21, in each second. Also, since we are talking about speed which is distance divided by time the word per means divided by. Ex.  $\frac{\text{miles}}{\text{hour}}$

The number 21 means that over a period of one second, the Porsche will change its speed by 21 miles per hour.

Per is used to mean a unit over a unit, or division, to signify rate.

21 means the miles the car will get you if you put one gallon of gasoline in the car. The word "per" means a ratio of miles in gallons of gas.

*Students have difficulty interpreting acceleration as the change in speed (in mph) that occurs in each second of the motion.*

- Tendency to focus on procedures rather than concepts
- Tendency to confuse related quantities
- Lack of functional understanding of “per”

Do students encounter difficulty when the ratio itself is unfamiliar (i.e., rather than the structure of the quantities)?

# Toy Car question

*A student studying motion conducts an experiment in which a windup toy car moves uniformly, traveling 60 cm in 2.4 sec. The student divides  $2.4/60$  and obtains 0.04.*

*Describe the information the number 0.04 gives about the motion and explain. If this number does not have a meaningful interpretation, state that explicitly.*

*Administered before, during and after instruction at a variety of institutions and levels of introductory physics*

# Toy Car question

## Correct student response

- Toy car moves 60 cm in 2.4 s.
- Interpret  $2.4/60 = 0.04$

Describe the information that the number 0.04 gives about the motion and explain your reasoning. If the number 0.04 does *not* have a meaningful interpretation in this situation, state that explicitly.

This tells you that it takes the car 0.04 seconds to move one cm.

# Toy Car question

## *Incorrect responses*

- *Toy car moves 60 cm in 2.4 s.*
- *Interpret  $2.4/60 = 0.04$*

*“0.04 is the distance in cm the car will move per second because the student divided the seconds by the centimeter.”*

*Tendency to interpret 0.04 as the speed of the car*

# Toy Car question

## *Incorrect responses*

- *Toy car moves 60 cm in 2.4 s.*
- *Interpret  $2.4/60 = 0.04$*

*“0.04 means cm/s. It is from distance/time = rate. The car is going 0.04 cm/sec.”*

*“0.04 could possibly represent the velocity?  $D/T = ?$  I know I’ve dealt with similar problems in math classes before, but I do not remember what this equation represents.”*

**Rote use of formula**

Results similar to Porsche question:

*Many students attend to surface features and concrete procedures, rather than the structure of the underlying reasoning.*

# Toy Car question

## *An additional response*

- Toy car moves 60 cm in 2.4 s.
- Interpret  $2.4/60 = 0.04$

*“Time over a distance is not useful for talking about the motion of the car. Dividing 60 cm/2.4 sec would give the average velocity and is far more useful. s/cm ... is not useful, unless you want to know how long it takes for the car to travel 1 cm, which is 0.04 sec.”*

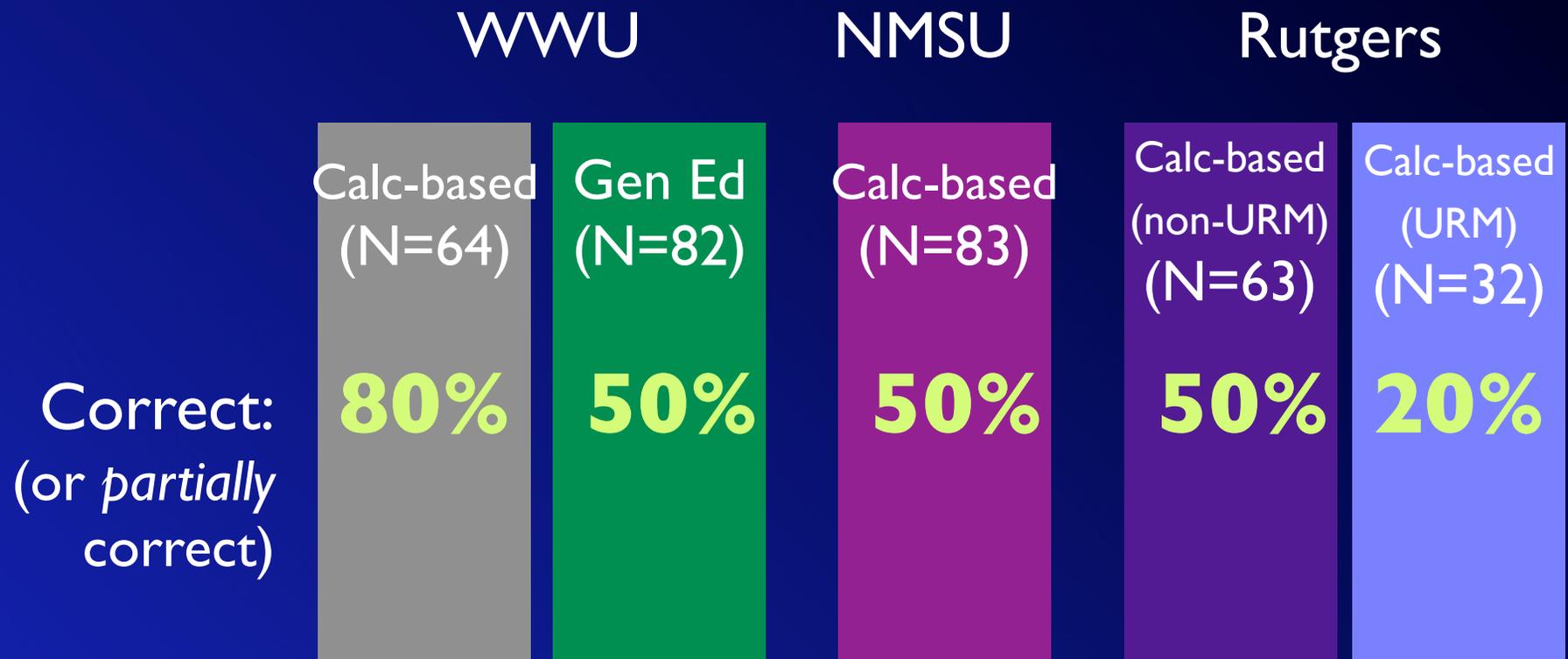
**Tendency to regard 0.04 s/cm as “meaningless”  
(even when student has interpreted it correctly)**

# Toy Car question

## Results

(before course instruction)

- Toy car moves 60 cm in 2.4 s.
- Interpret  $2.4/60 = 0.04$



# Zone of Mathematization

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Concepts in the introductory course are well within a physicist's **zone of mathematization**, but are beyond or just at the edge of students' zones.

*Instructors may have forgotten what it's like to operate outside the zone.*

*Let me remind you.*

A motion is characterized by  $9.8 \text{ m/s}^2$ .

Provide an *interpretation* of the number 9.8.

A motion is characterized by  $0.1 \text{ s}^2/\text{m}$ .

Provide an *interpretation* of the number 0.1.

### “Inverse g task” given informally to physics faculty

*“Is this pedagogically speaking, or physically? Well, let’s see. It’s the... the reciprocal of the speed is the time it takes to cover a certain amount of ground. So this is the time it takes to cover that certain amount of ground, and then the time it takes to do that next chunk of that, so, I don’t know what that means physically ... but that’s what I would characterize it as.”*

# Manipulating task difficulty to explore the variability/stability of student reasoning

**Structure**



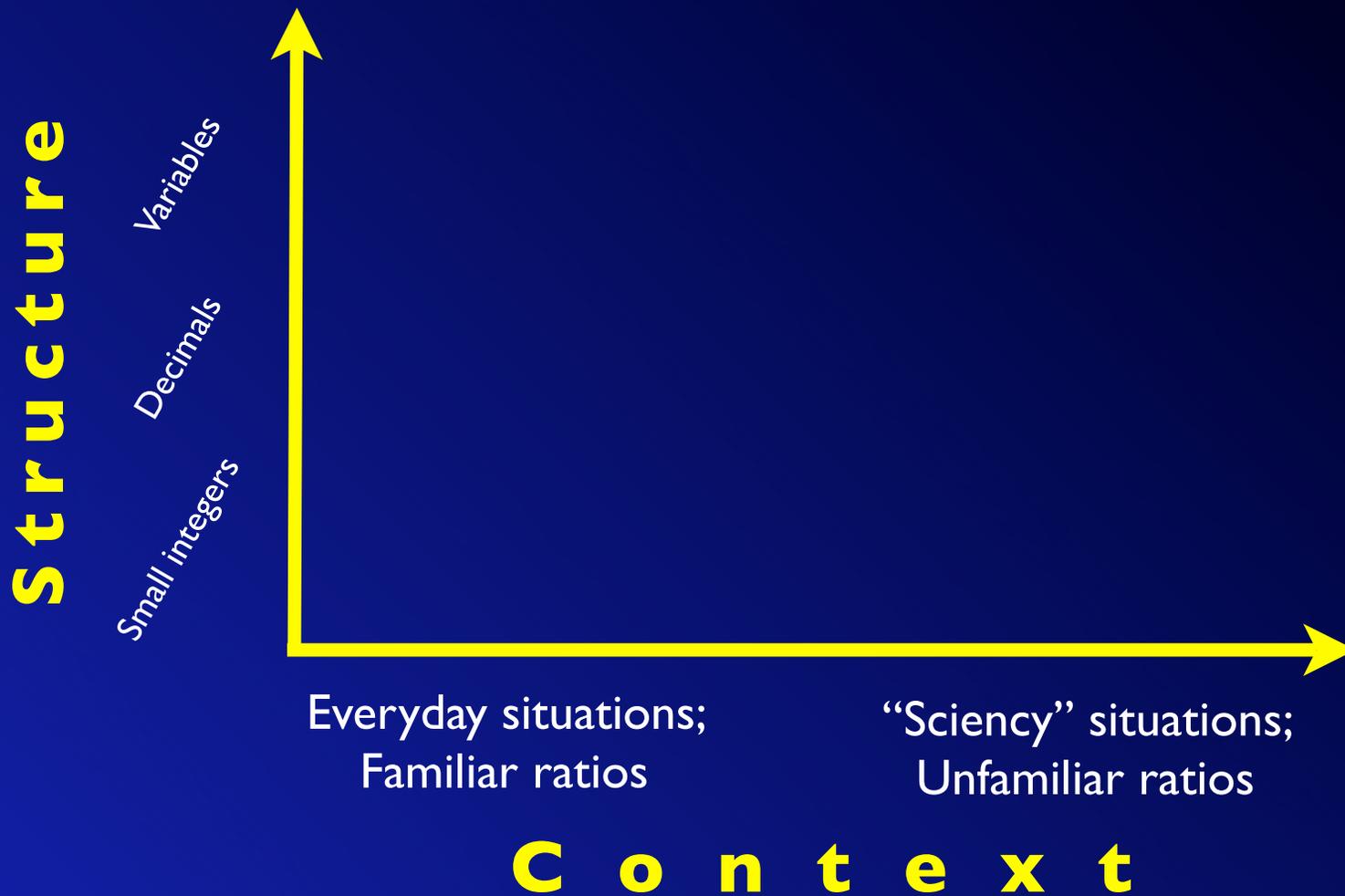
Everyday situations;  
Familiar ratios

“Sciency” situations;  
Unfamiliar ratios

**C o n t e x t**

## Research question:

*To what extent do quantity structure and physical context impact student facility with the verbal interpretation of ratios?*



# Paired Questions Study

- Suite of 10 MC questions given in mechanics, E&M, and chemistry courses at Rutgers University ( $N > 1000$ )
- Suite administered with FCI, CSEM, and CCI at beginning and end of course
- Paired questions administered at random to half of each participating class
- Over 15 versions of the suite used; total of 30 distinct items

## Paired Questions: *Verbal Interpretation of Ratio*

Catherine is hired to paint the ceiling of her aunt's living room. She covers the ceiling with a uniform coat of paint. The ceiling has a surface area of 580 square feet. After finishing, Catherine notes that she used 2.4 gallons of paint. Catherine divides 580 by 2.4 and gets 241.7.

Which of the following statements about the number 241.7 is true?

- a. 241.7 is the total number of gallons of paint used
- b. 241.7 is the total number of square feet of surface area covered by the paint
- c. 241.7 is the number of gallons of paint that covers one square foot
- d. 241.7 is the number of square feet that one gallon of paint covers
- e. none of the above

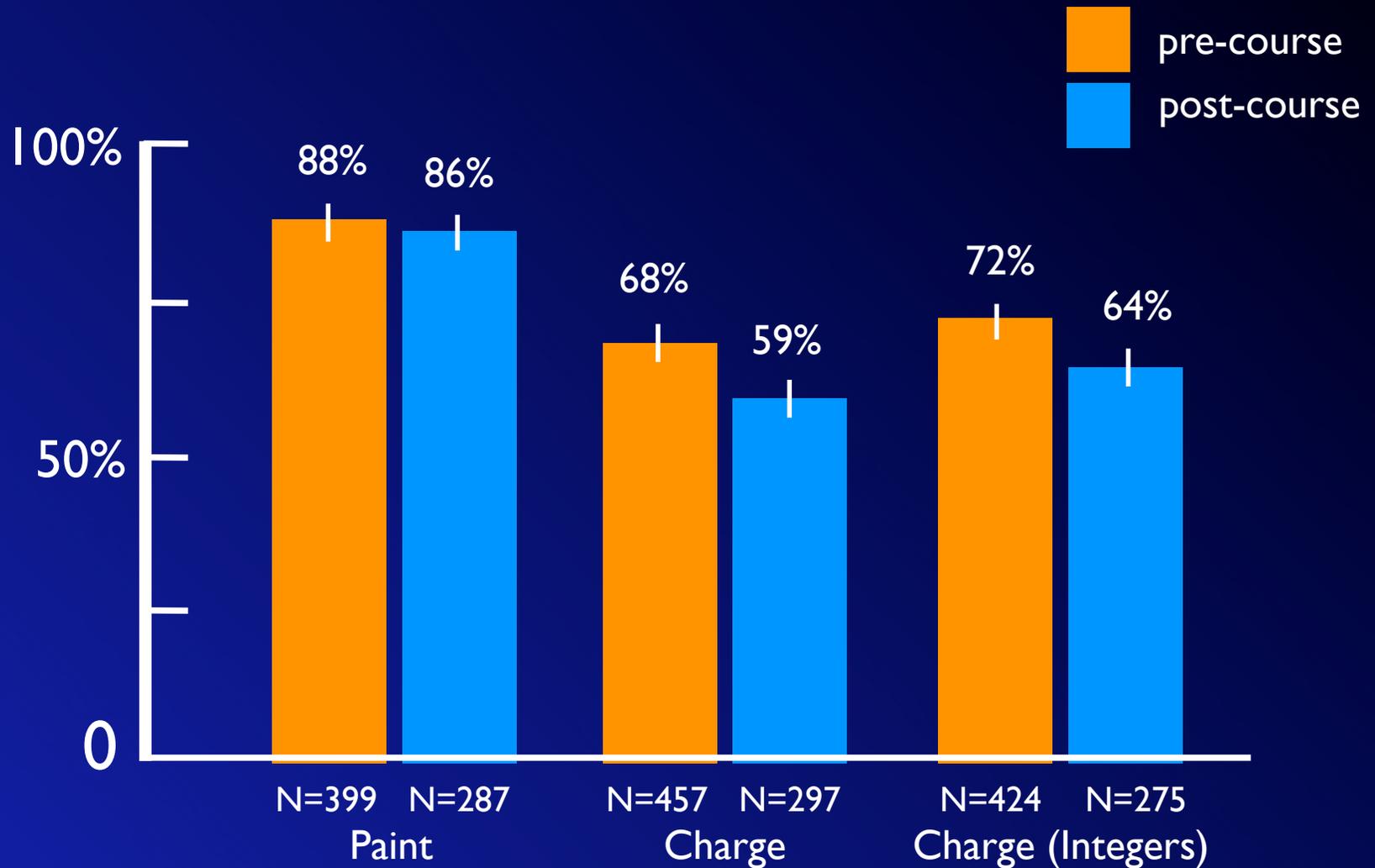
Catherine shuffles her feet across her living room carpet and then she touches a doorknob, which has a surface area of 580 square centimeters. When she touches the doorknob she transfers 2.4 ~~microcoulombs~~ of electric charge that spreads out uniformly over the doorknob's surface.

Catherine divides 580 by 2.4 and gets 241.7.

Which of the following statements about the number 241.7 is true?

- a. 241.7 is the total number of ~~microcoulombs~~ of charge transferred
- b. 241.7 is the total number of square centimeters of surface area covered by the charge
- c. 241.7 is the number of ~~microcoulombs~~ of charge that covers one square centimeter
- d. 241.7 is the number of square centimeters that one ~~microcoulomb~~ of charge covers
- e. none of the above

# Paired Questions: *Verbal Interpretation of Ratio*



## Summary:

*Verbal interpretation of ratios is challenging across contexts*

- Students tend to engage with algorithms and procedures rather than sense-making
- Students lack of functional understanding of “per”
- Reasoning facility appears to decrease as physical context becomes less familiar
- Instruction may not automatically promote fluency

*Use of interviews to explore how deeply  
student difficulties are held  
(or, conversely, how readily students can  
access productive resources)*

# Interview 1

(Preservice elementary teacher)

**S1:** *“... the, uh, .04 describes, um, how many centimeters the car traveled in each second...*

*...from my experience you don't do it that way. It's usually the distance over the time. So it should be 60 cm over 2.4 seconds equals whatever. And then it would be, um, centimeters per second. And that's a more meaningful answer I think.”*

**I:** *“More meaningful in what way, can you tell me?”*

**S1:** *“I'm not sure, like, scientifically why it would be more meaningful. But . . . the distance formula is equal to rate over time. I don't know, does that help me? [Writing. Inaudible.] I don't know if I want to go that way. **That's what I see more often.**”*

# Interview 1

*(Preservice elementary teacher)*

**I:** *“Tell me again about, about 0.04, what information it gives us.”*

**S1:** *“The student divided the seconds by centimeters. So that gave us how far the car moved, um. Oh, gosh. Seconds per centimeter. How far it took, or, how long it took the car to travel one centimeter. I think. Seconds per centimeter. Yes.”*

**I:** *“Okay. How do you think about the word ‘per’ in that phrase, ‘seconds per centimeter’?”*

**S1:** *“For every sec-, or, [pause]. It describes, um, how much, like, if you ha-, um. [Pause.] My brain doesn’t want to work anymore.”*

# Interview 1

*(Preservice elementary teacher)*

Substantial cognitive load associated with  
interpretation of unfamiliar ratio

# Interview 2

(Preservice elementary teacher)

**S2:** *“It doesn’t have a meaningful interpretation because it’s not something we look at. Like the opposite.”*

**I:** *“What could .04 be telling us?”*

**S2:** *“So she did 2.4 seconds divided by 60 cm. So that would give us .04 seconds per centimeter. So if you wanted the, to know, like, how, how fast ... **how fast it moved in one centimeter.**”*

**I:** *“Tell me more about what you mean.”*

**S2:** *“if it was centimeters per second, that would be the speed of it, so it would move, that many, like a certain amount of centimeters in one second.”*

*“But since she did 2.4 seconds divided by 60 cm, then that gives her how, that gives her 0.04, um, seconds in one centimeter. So it tells you how much, or **how far it moves in 1 cm.**”*

# Interview 2

(Preservice elementary teacher)

**S2:** “Or, not how far. How, **how long it takes it to move one centimeter.**”

**I:** “Okay. So if there was another toy car, but the number that came out was 0.05. Would that mean the second toy car was faster or slower than this one?”

**S2:** “Too many fractions. So, um, trying to think if 0.04 seconds is faster or slower than 0.05 seconds. [Long pause] “

**I:** “Can you tell me what’s tricky about that?”

**S2:** “Yeah, fractions.”

...

**S2:** “I don’t know. I can’t, I don’t know. I can’t think. **It depends on what number is smaller or bigger and I don’t know.**”

# Summary of S2 interpretation of 0.04 s/cm

*Not meaningful because unfamiliar*



*How fast car moves in 1 cm*



*How far car moves in 1 cm*



*How long car takes to move 1 cm*



*Struggle to compare the motions  
represented by 0.04 and 0.05*

## Interview 3

*(Math major in calc-based physics)*

*“So,  $2.4/60$  reduces to the fraction  $.04/1$ . And so divided by it means ‘per’, which means during a distance interval, I guess. It’s hard to say. Um, for 0.04 seconds, it travels 1 cm. It’s much harder to describe it when it’s not the way it’s used to being presented.”*

**Student reasoning difficulties with the verbal interpretation of ratios are in some cases not easily resolved**

Need: Develop a more comprehensive picture of student reasoning about ratio and proportion.

Response: Ask students to reverse the line of thought.

- Instead of interpreting a given ratio,
- construct a ratio to match a desired interpretation.



Interpreting a given ratio



Constructing a desired ratio

*Articulating an additional “mode” of proportional reasoning*



Bobbing block question:

*B bobs occur in 10 seconds.  
How many sec for a single bob?*

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Interview response:

*“It’s hard to go backwards. It’s more intuitive to think of doing something per unit of time than it is to think of the amount of time required to do something ...”*

*In the physics class that I’m in now, having the period and the frequency, relating those two, like, I know that there’s, like, [makes air quotes] an equation for it. Frequency is one over period. But thinking about it in an actual way, trying to represent it physically, takes a lot of mental effort for me personally.”*

Bartholomew is making rice pudding using his grandmother's recipe. For three servings of pudding the ingredients include 0.75 pints of milk and 0.5 cups of rice. Bartholomew looks in his refrigerator and sees he has one pint of milk. Given that he wants to use all of the milk, which of the following expressions will help Bartholomew figure out how many cups of rice he should use?

$0.5/0.75$

$0.75/0.5$

$0.5 \times 0.75$

$(0.5 + 1) \times 0.75$

none of these

*Similar difficulties are observed on a variety of constructing ratio tasks*

## Results from Paired Questions study

	Underprepared (n=114, s.d.≈ 9%)		Mainstream (n=624, s.d.≈ 4%)	
	<i>Pre(%)</i>	<i>Post(%)</i>	<i>Pre(%)</i>	<i>Post(%)</i>
Rice _integer	60	63	84	79
Rice _decimal	32	61	66	69
Rice _variable	27	20	47	38

Strong influence of structure on student performance  
*Use of abstract representation interferes with reasoning*

Construct ratio from  
measured values

Interpret ratio  
(using “for each”)

**Proportional  
Reasoning**

```
graph TD; A([Proportional Reasoning]) --> B[Interpret ratio (using "for each")]; A --> C[Construct ratio from measured values];
```

# Operationalizing Proportional Reasoning

*Systematic progress in understanding student thinking and promoting student learning is facilitated by explicit articulation of the modes of reasoning we expect students to master.*

Construct ratio from  
measured values

Interpret ratio  
(using “for each”)

Apply ratio to  
calculate/predict

## Proportional Reasoning

Select ratio  
as appropriate measure

Scale functional  
relationships

Translate  
representations

Disclaimer:

These specific modes of reasoning are likely *not* employed explicitly by students

and are *not* necessarily distinct cognitive entities.

# Promoting reasoning fluency using Invention Tasks

*(Approach based on work of Schwartz et al)*

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An invention task challenges students to identify the key features of a system . . .

and develop a way to characterize the system that allows meaningful comparisons to be made.

# Invention Instruction

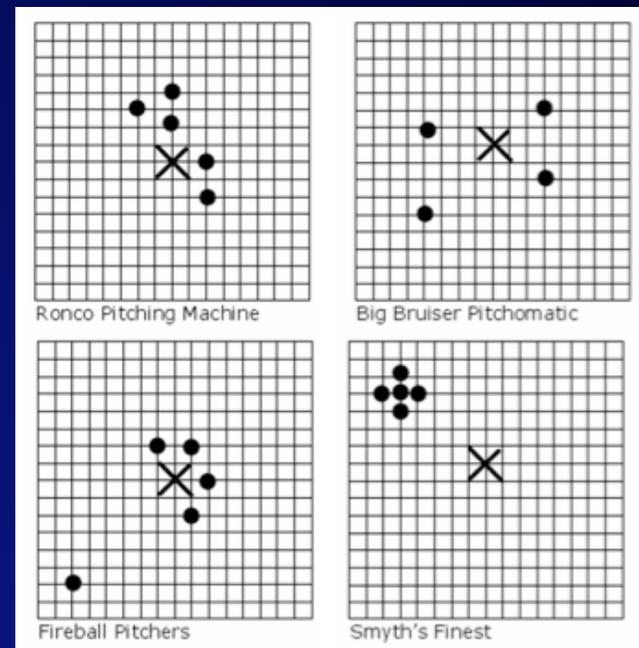
Often uses *contrasting cases*:

- situations that are recognizably similar,
- yet different in important ways.

*Example:*

Evaluating the reliability of different pitching machines.

From Himmelberger and Schwartz, 2000



Invention instruction:  
*Preparation for future learning*

*At Rutgers, WWU, and NMSU, we are adapting invention instruction to introductory physics courses, with a specific goal of developing student facility with ratio and product quantities in context.*

# Strategy for implementing invention instruction

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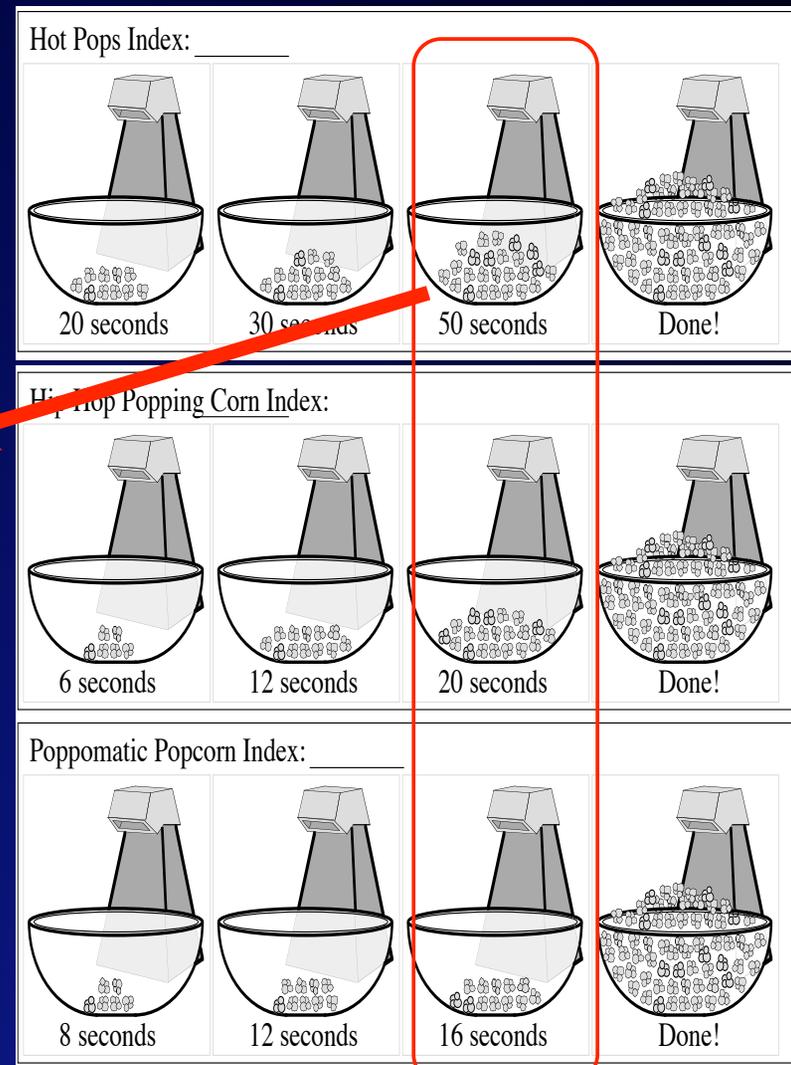
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- Develop tasks in which students invent quantities to characterize physical systems.
- Group tasks into sequences that bridge everyday to abstract contexts.
- Integrate questions to support development of reasoning facility.

# Sample Invention Sequence: *Kinematics*

Invent a *popping index* to rank the different brands according to how fast they pop.

**“Hmmm, Hot Pops has the most corns but also takes the longest...”**



# Sample Invention Sequence: *Kinematics*

Invent a *popping index* to rank the different brands according to how fast they pop.

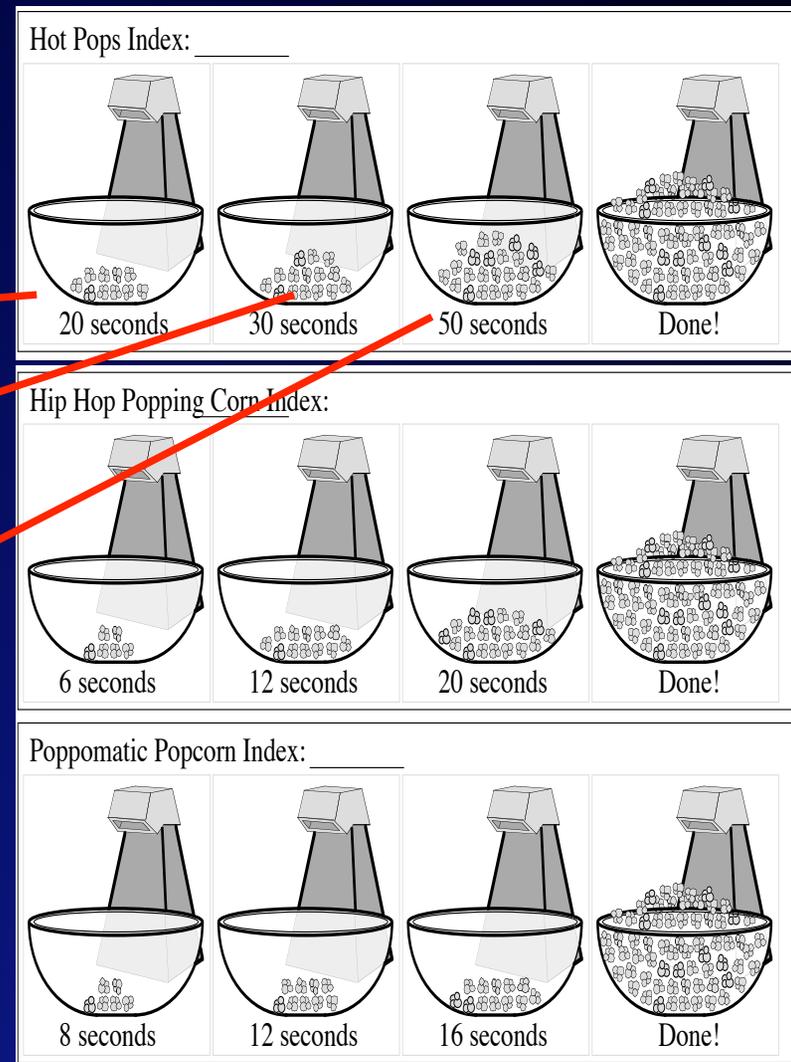
“10 corns in 20 sec...”

“5 more corns in the next 10 sec...”

“and 10 more in the next 20 sec...”

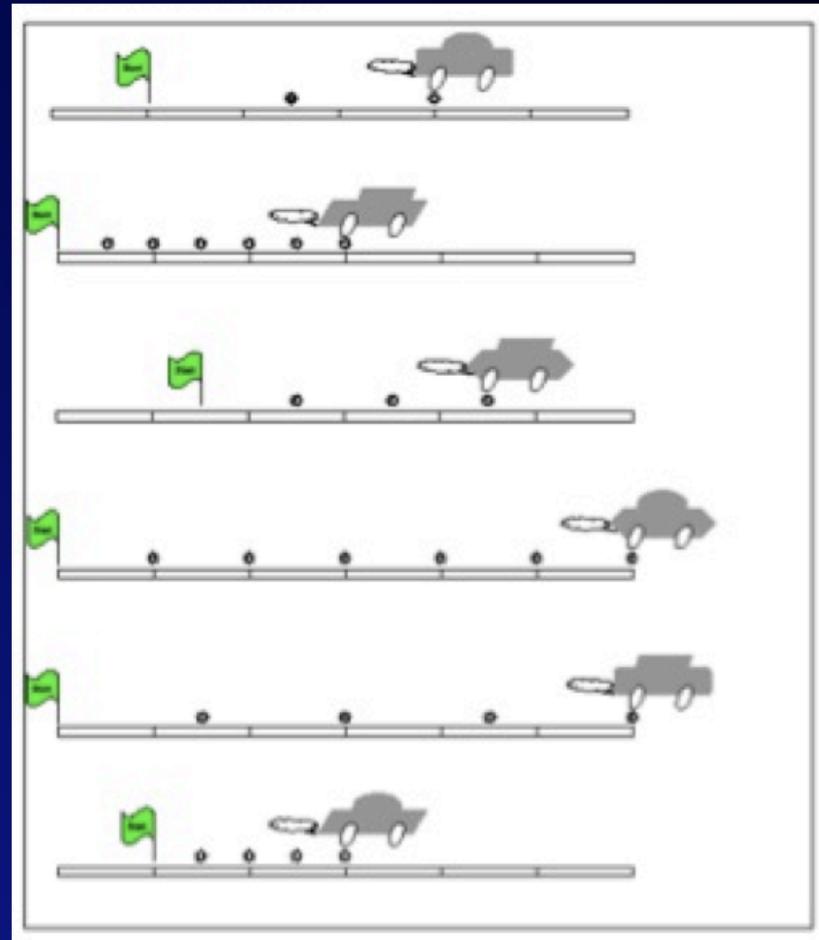
“I see a pattern! Let's invent

$$\frac{(\# \text{ corns popped})}{(\text{time elapsed})}$$
 as our index!”

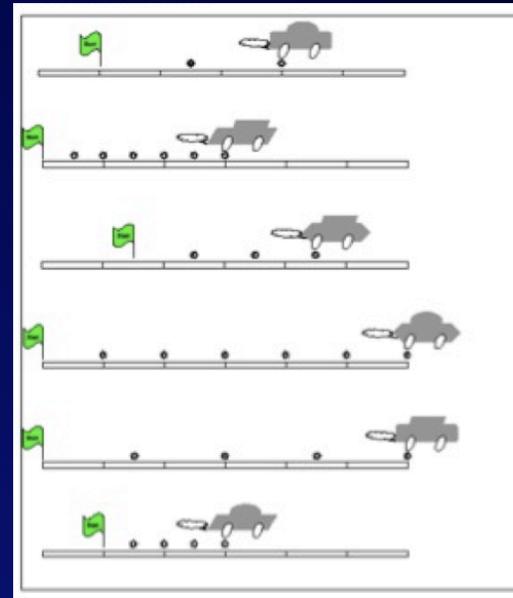
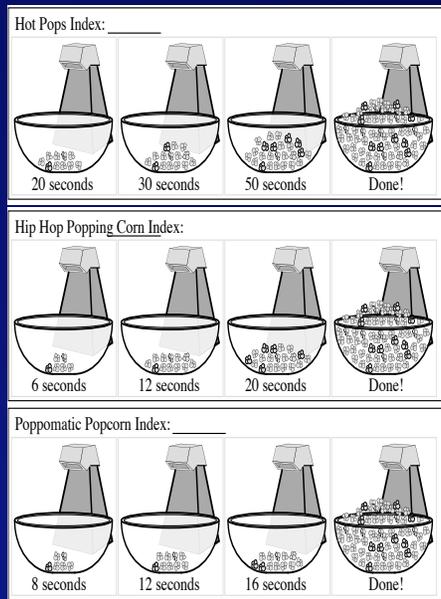


## Sample Invention Sequence: *Kinematics*

Let's look at another index. Come up with a *fastness* index for cars. Each car drips oil once every second. Cars from the same company have the same fastness. How many companies?



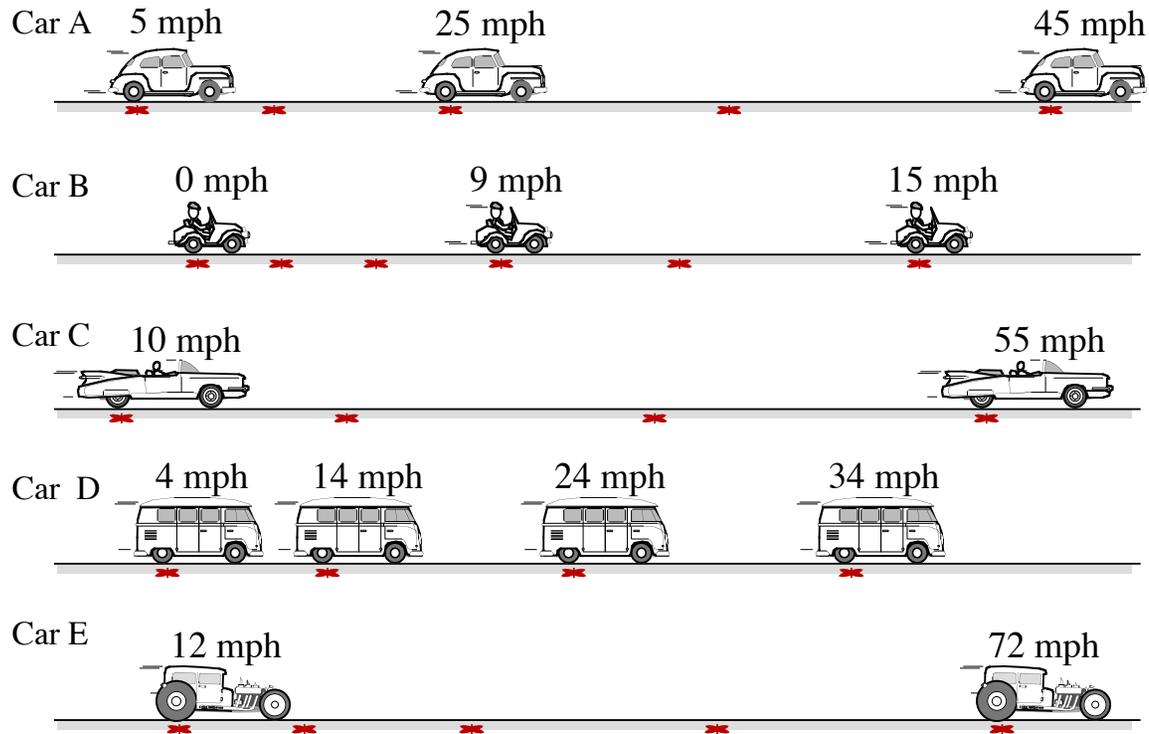
# Sample Invention Sequence: *Kinematics*



A full bowl of popcorn has 60 corns. Determine the time for the fastest popcorn to fill a bowl.

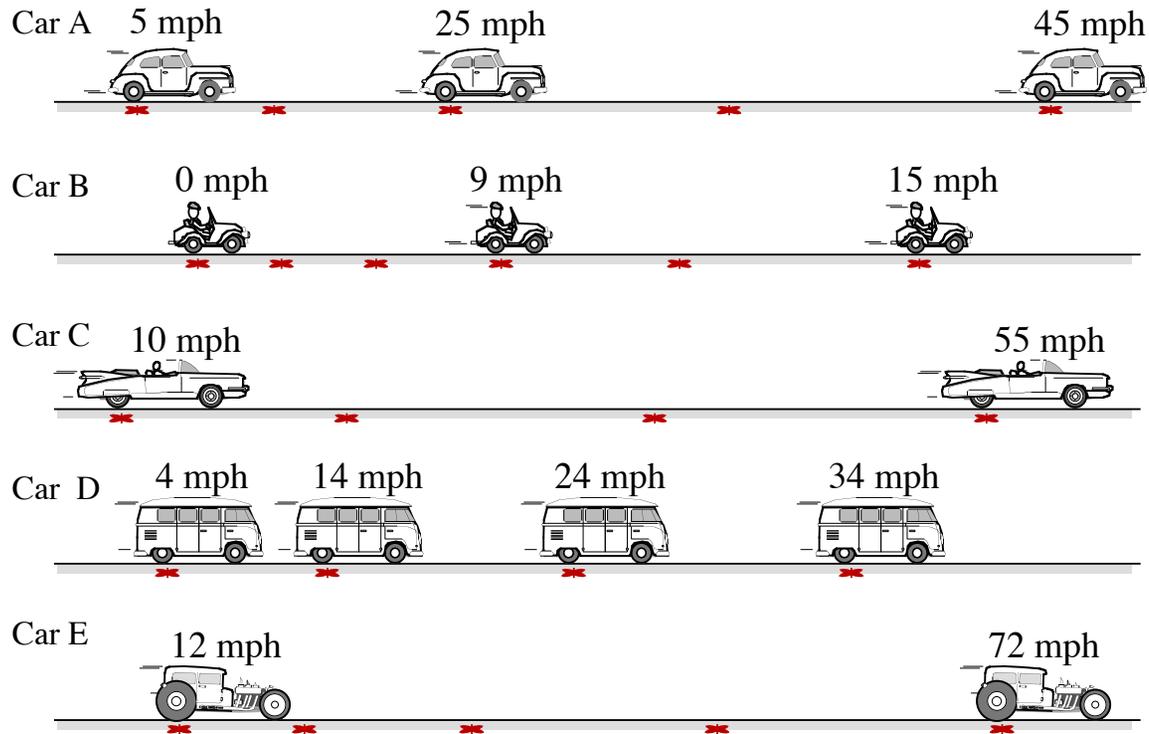
Write an expression for the time required for the fastest car to travel  $B$  blocks.

# Sample Invention Sequence: *Kinematics*



Now come up with a *speeding up index* for cars with dripping oil.

# Sample Invention Sequence: *Kinematics*



Explain the difference between *moving fast* and *speeding up quickly*.

## Assessing student learning: *Toy car task*

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Before course instruction:

**42% correct**

After course instruction using invention sequences:

**> 95% correct**

N = 84, Gen Ed Physics at WWU

## Assessing student learning: *Inverse g task*

[5 pts] An object released near the surface of the Earth is found to move with constant acceleration, speeding up at a rate of about  $10 \text{ m/s}^2$ . Consider the inverse of this number:  $0.1 \text{ s}^2/\text{m}$ . Give an interpretation of this number. Using everyday language, describe the specific information that 0.1 provides about the object's motion.

$$\frac{10 \text{ m/s}}{\text{second}} \quad \text{or} \quad \frac{.1 \text{ Second}}{1 \text{ m/s}}$$

This number tells us that for every .1 second that passes the object increases its speed by 1 m/s.

**30% correct at end of course**

Gen Ed Physics student:

*“The invention activities helped me to understand how the core of science works...values presented by scientists aren't some sort of godly measurements that are perfect and unassailable, and helped to ‘let us in’ on how physics really works; it allowed us to understand the process and once we understood the process we could create measurement relationships of our own as well as understand those of others.”*

Inservice HS physics teacher (after using Invention materials with his students):

*“I did the following problem as an example in lecture of constant velocity with more than one object for the umpteenth time in my life:*

The first two sprinters in a 100 m race finish in 10.5 s and 11.5 s. Assuming constant speed, what is the separation between them when the first one crosses the finish line?

*“After having gone through a standard analytical and then a graphical solution in lecture, a student came up to me after lecture and told me he thought of a really simple way to do that problem. He said:*

*‘The second runner runs 1 second less than the time necessary for him to run 100 m when the first runner crosses the line. So, if you find the 2nd runner's fastness index, then its value tells you how far he'll be behind the 100m mark one second before he gets there.’*

<http://inventiontasks.physics.rutgers.edu/index.html>